(q, t)-Laguerre polynomials

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The Laguerre polynomials

For $\alpha > -1$ and $m$ a nonnegative integer, Laguerre’s equation is the linear second order ODE

$$x y''(x) + (\alpha + 1 - x)y'(x) + m y(x) = 0$$

Its polynomial solutions, denoted by $L_m^{(\alpha)}(x)$, are known as the (generalised/associated) Laguerre polynomials.
The first few Laguerre polynomials are given by

\[ L_0^{(\alpha)}(x) = 1 \]

\[ L_1^{(\alpha)}(x) = -x + \alpha + 1 \]

\[ L_2^{(\alpha)}(x) = \frac{1}{2}x^2 - (\alpha + 2)x + \frac{1}{2}(\alpha + 1)(\alpha + 2) \]
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and more generally one has

\[ L_m^{(\alpha)}(x) = \sum_{i=0}^{m} \binom{m + \alpha}{m - i} \frac{(-x)^i}{i!} \]
It is easily verified that the Laguerre polynomials satisfy a three-term recurrence relation:

$$(m + 1)L_{m+1}^{(\alpha)}(x) = (2m + 1 + \alpha - x)L_m^{(\alpha)}(x) - (m + \alpha)L_{m-1}^{(\alpha)}(x)$$

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The orthogonalising measure \(\mu\) for the Laguerre polynomials is given by

\[d\mu(x) = x^\alpha e^{-x}dx\]

supported on the positive half-line:

\[\int_0^\infty L_m^{(\alpha)}(x)L_n^{(\alpha)}(x) d\mu(x) = \delta_{mn} \frac{\Gamma(m + \alpha + 1)}{m!}\]
The orthogonality with respect to the Laguerre measure $\mu$ may be proved as follows:

- Laguerre’s equation is equivalent to the statement that $L_m^{(\alpha)}(x)$ is the eigenfunction with eigenvalue $m$ of the second order differential operator

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The orthogonality with respect to the Laguerre measure \( \mu \) may be proved as follows:

- Laguerre’s equation is equivalent to the statement that \( L^{(\alpha)}_m(x) \) is the eigenfunction with eigenvalue \( m \) of the second order differential operator

\[
\mathcal{L} = -x \frac{d^2}{dx^2} + (x - \alpha - 1) \frac{d}{dx}
\]

- The operator \( \mathcal{L} \) is self-adjoint with respect to the inner product

\[
\langle f, g \rangle = \int_0^\infty f(x)g(x)d\mu(x)
\]
i.e.,

\[
\langle \mathcal{L}f, g \rangle = \langle f, \mathcal{L}g \rangle
\]
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- **Differential equation** $q$-difference equation
- **Integration** Jackson-integration
Let $\mathcal{D}_q$ be the $q$-derivative operator

$$(\mathcal{D}_q f)(x) = \frac{f(xq) - f(x)}{(q - 1)x}$$

If $f$ is differentiable at $x$ then

$$\lim_{q \to 1} (\mathcal{D}_q f)(x) = f'(x)$$

For example,

$$\mathcal{D}_q x^m = [m] x^{m-1}$$

with $[a]$ the $q$-number $[a] = (1 - q^a)/(1 - q)$.
The $q$-Laguerre polynomial $L_m^{(\alpha)}(x) = L_m^{(\alpha)}(x; q)$ is the eigenfunction, with eigenvalue $[m]$, of the $q$-difference operator

\[ \mathcal{L} = (x + 1) \mathcal{D}_q - q^{-\alpha-1} \mathcal{D}_{q^{-1}} \]
The $q$-Laguerre polynomial $L_m^{(\alpha)}(x) = L_m^{(\alpha)}(x; q)$ is the eigenfunction, with eigenvalue $[m]$, of the $q$-difference operator

$$\mathcal{L} = (x + 1)D_q - q^{\alpha-1}D_{q^{-1}}$$

The first few $q$-Laguerre polynomials are given by

$$L_0^{(\alpha)}(x) = 1$$

$$L_1^{(\alpha)}(x) = -q^{\alpha+1}X + [\alpha + 1]$$

$$L_2^{(\alpha)}(x) = \frac{1}{[2]}X^2 - q^{\alpha+1}[\alpha + 2]X + \frac{1}{[2]}[\alpha + 1][\alpha + 2]$$

where $X = x/(1 - q)$. 
Again there is a three-term recurrence:

\[
[m + 1]L_{m+1}^{(\alpha)}(x)
= \left([m + 1] + q[m + \alpha] - q^{2m+\alpha+1}(1 - q)x\right)L_m^{(\alpha)}(x)
- q[m + \alpha]L_{m-1}^{(\alpha)}(x)
\]

Hence there exists an orthogonalising measure \( \mu \) such that the \( q \)-Laguerre polynomials form an orthogonal family.

To describe this measure we need the Jackson integral.
The Jackson integral over the positive reals is defined as

\[
\int_0^\infty f(x) \, dqx = (1 - q) \sum_{n=-\infty}^{\infty} f(q^n)q^n
\]
Because of its discreteness something is lost, and there is no $q$-analogue of
\[ c \int_0^\infty f(cx) \, dx = \int_0^\infty f(x) \, dx \quad c > 0 \]

Hence we also define
\[ \int_0^{c \cdot \infty} f(x) \, d_q x = (1 - q) \sum_{n=-\infty}^{\infty} f(cq^i) cq^i \]
The $q$-Laguerre polynomials satisfy the discrete orthogonality

$$
\int_0^{c \cdot \infty} L_m^{(\alpha)}(x)L_n^{(\alpha)}(x) d\mu(x)
$$

$$
= \delta_{mn} [c(1 - q)]^{\alpha + 1} q^{-m} \frac{\Gamma_q(m + \alpha + 1)}{\Gamma_q(m + 1)} \frac{\theta(-acq)}{\theta(-c)}
$$

where

$$
d\mu(x) = x^\alpha e_q(-x) d_q x
$$

and

$$
e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]!} = \frac{1}{(1 - x)(1 - xq)(1 - xq^2) \cdots}
$$

is the $q$-exponential function.
The previous results can again be restated as follows:

- The $q$-Laguerre polynomial $L_m^{(\alpha)}(x)$ is the eigenfunction with eigenvalue $[m]$ of

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$$\mathcal{L} = (x + 1)\mathcal{D}_q - q^{-\alpha - 1}\mathcal{D}_{q^{-1}}$$

- The operator $\mathcal{L}$ is self-adjoint with respect to the inner product

$$\langle f, g \rangle = \int_0^{c \cdot \infty} f(x)g(x)d\mu(x)$$

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The \((q, t)\)-Laguerre polynomials

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**Trivial example.** The classical Laguerre polynomials

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\langle 1, 1 \rangle = \int_0^\infty d\mu(x) = \int_0^\infty x^\alpha e^{-x} dx = \Gamma(\alpha + 1)
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**Non-trivial example.** The Lie algebra pair \((A_{n-1}, A_1)\)

\[
\int_{[0, \infty)^n} \prod_{i=1}^n x_i^\alpha e^{-x_i} \prod_{1 \leq i < j \leq n} |x_i - x_j|^{2\gamma} dx_1 \cdots dx_n = \prod_{i=0}^{n-1} \frac{\Gamma(\alpha + 1 + i\gamma)\Gamma(1 + (i + 1)\gamma)}{\Gamma(1 + \gamma)}
\]
For \( x = (x_1, \ldots, x_n) \) let \( \partial_{q,i} \) be the partial difference operator

\[
(\partial_{q,i} f)(x) = \frac{f(x_1, \ldots, x_{i-1}, qx_i, x_{i+1}, \ldots, x_n) - f(x)}{(q - 1)x_i}
\]

Define the operators \( E_r(q, t) \) acting on \( \Lambda_n = \mathbb{Q}(q, t)[x_1, \ldots, x_n] \mathcal{S}_n \) by

\[
E_r(q, t) = \sum_{i=1}^{n} x_i^r \left( \prod_{j=1}^{n} \frac{tx_i - x_j}{x_i - x_j} \right) \partial_{q,i}
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Note that for $n = 1$

$$E_0(q, t) = \mathcal{D}_q \quad \text{and} \quad E_1(q, t) = x \mathcal{D}_q$$
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Lemma. For \( r \geq 0, E_r : \Lambda_n \to \Lambda_n \)
Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition (a weakly decreasing sequence of nonnegative integers) and $\mathcal{L} : \Lambda_n \rightarrow \Lambda_n$ the operator

$$\mathcal{L} = E_1(q, t) + E_0(q, t) - q^{-\alpha-1}E_0(q^{-1}, t^{-1})$$

The eigenfunction of $\mathcal{L}$ with eigenvalue

$$t^{n-1}[\lambda_1] + \cdots + t[\lambda_{n-1}] + [\lambda_n]$$

defines the $(q, t)$-Laguerre polynomial $L^{(\alpha)}_\lambda(x) = L^{(\alpha)}_\lambda(x; q, t)$. 
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**Lemma.** The $L^{(\alpha)}_\lambda(x)$ form a basis of $\Lambda_n$. 
For \((q, t) = (q, q^\gamma)\) with \(q \to 1\) the \((q, t)\)-Laguerre polynomials were studied by
Let \( t = q^\gamma \) and define an inner product on \( \Lambda_n \) by

\[
\langle f, g \rangle = \int_{x_1=0}^{\infty} \cdots \int_{x_n=0}^{tx_{n-1}} f(x)g(x)d\mu(x)
\]

where

\[
d\mu(x) = \prod_{1 \leq i < j \leq n} x_i^{2\gamma} \left( q^{1-\gamma} x_j/x_i \right)^\gamma \left( x_j/x_i \right)^\gamma \prod_{i=1}^{n} x_i^\alpha e_q(-x_i) d_q x_i
\]

and

\[
(a)_z = \frac{(1 - a)(1 - aq)(1 - aq^2) \cdots}{(1 - aq^z)(1 - aq^{z+1})(1 - aq^{z+2}) \cdots}
\]
Theorem. The operator $\mathcal{L}$ is self-adjoint with respect to the inner product on $\Lambda_n$. 
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**Theorem** $c_{\lambda}$ can explicitly be computed in terms of “nice” special functions, such as theta functions and $q$-Gamma functions.
Ingredients in proofs are:

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The End