(q, t)-Laguerre polynomials

Ole Warnaar

School of Mathematics and Physics

The Laguerre polynomials

For $\alpha > -1$ and m a nonnegative integer, Laguerre's equation is the linear second order ODE

$$x\,y''(x) + (\alpha + 1 - x)y'(x) + m\,y(x) = 0$$

Its polynomial solutions, denoted by $L_m^{(\alpha)}(x)$, are known as the (generalised/associated) Laguerre polynomials.

The first few Laguerre polynomials are given by

$$L_0^{(\alpha)}(x) = 1$$

$$L_1^{(\alpha)}(x) = -x + \alpha + 1$$

$$L_2^{(\alpha)}(x) = \frac{1}{2}x^2 - (\alpha + 2)x + \frac{1}{2}(\alpha + 1)(\alpha + 2)$$

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and more generally one has

$$L_m^{(\alpha)}(x) = \sum_{i=0}^{m} \binom{m+\alpha}{m-i} \frac{(-x)^i}{i!}$$

It is easily verified that the Laguerre polynomials satisfy a three-term recurrence relation:

$$(m+1)L_{m+1}^{(\alpha)}(x) = (2m+1+\alpha-x)L_m^{(\alpha)}(x) - (m+\alpha)L_{m-1}^{(\alpha)}(x)$$

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The orthogonalising measure μ for the Laguerre polynomials is given by

$$\mathrm{d}\mu(x) = x^{\alpha}\mathrm{e}^{-x}\mathrm{d}x$$

supported on the positive half-line:

$$\int_0^{\infty} L_m^{(\alpha)}(x)L_n^{(\alpha)}(x)\,\mathrm{d}\mu(x) = \delta_{mn}\,\frac{\Gamma(m+\alpha+1)}{m!}$$

The orthogonality with respect to the Laguerre measure μ may be proved as follows:

- Laguerre's equation is equivalent to the statement that $L_m^{(\alpha)}(x)$ is the eigenfunction with eigenvalue m of the second order differential operator

$$\mathcal{L} = -x\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (x - \alpha - 1)\frac{\mathrm{d}}{\mathrm{d}x}$$

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- The operator $\mathcal{L}$ is self-adjoint with respect to the inner product

$$\langle f, g \rangle = \int_0^\infty f(x)g(x)\mathrm{d}\mu(x)$$

i.e.,

$$\langle \mathcal{L}f, g \rangle = \langle f, \mathcal{L}g \rangle$$

The q-Laguerre polynomials

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- ~~Differential equation~~ q-difference equation
- ~~Integration~~ Jackson-integration

Let $\mathscr{D}_q$ be the q-derivative operator

$$(\mathscr{D}_q f)(x) = \frac{f(xq) - f(x)}{(q-1)x}$$

If f is differentiable at x then

$$\lim_{q \to 1} (\mathscr{D}_q f)(x) = f'(x)$$

For example,

$$\mathscr{D}_q x^m = [m]\, x^{m-1}$$

with $[a]$ the q-number $[a] = (1 - q^a)/(1 - q)$.

The q-Laguerre polynomial $L_m^{(\alpha}(x) = L_m^{(\alpha}(x; q)$ is the eigenfunction, with eigenvalue $[m]$, of the q-difference operator

$$\mathcal{L} = (x+1)\mathscr{D}_q - q^{-\alpha-1}\mathscr{D}_{q^{-1}}$$

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The first few q-Laguerre polynomials are given by

$$L_0^{(\alpha)}(x) = 1$$

$$L_1^{(\alpha)}(x) = -q^{\alpha+1}X + [\alpha+1]$$

$$L_2^{(\alpha)}(x) = \frac{1}{[2]}X^2 - q^{\alpha+1}[\alpha+2]X + \frac{1}{[2]}[\alpha+1][\alpha+2]$$

where $X = x/(1-q)$.

Again there is a three-term recurrence:

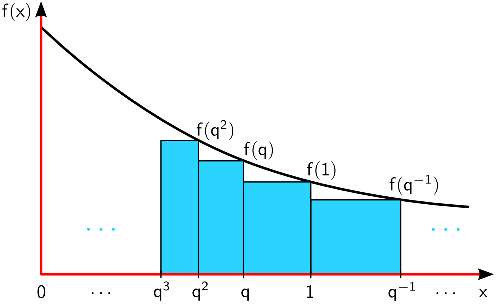
$$[m+1]L_{m+1}^{(\alpha)}(x) = \Big([m+1] + q[m+\alpha] - q^{2m+\alpha+1}(1-q)x\Big)L_m^{(\alpha)}(x) - q[m+\alpha]L_{m-1}^{(\alpha)}(x)$$

Hence there exists an orthogonalising measure μ such that the q-Laguerre polynomials form an orthogonal family.

To describe this measure we need the Jackson integral.

The Jackson integral over the positive reals is defined as

$$\int_0^{\infty} f(x) \mathrm{d}_q x = (1-q) \sum_{n=-\infty}^{\infty} f(q^i) q^i$$



Because of its discreteness something is lost, and there is no q-analogue of

$$c\int_0^\infty f(cx)\mathrm{d}x = \int_0^\infty f(x)\mathrm{d}x \qquad c > 0$$

Hence we also define

$$\int_0^{c\cdot\infty} f(x)\mathrm{d}_q x = (1-q)\sum_{n=-\infty}^{\infty} f(cq^i)cq^i$$

The q-Laguerre polynomials satisfy the discrete orthogonality

$$\int_0^{c\cdot\infty} L_m^{(\alpha)}(x) L_n^{(\alpha)}(x) \mathrm{d}\mu(x) = \delta_{mn}[c(1-q)]^{\alpha+1} q^{-m} \frac{\Gamma_q(m+\alpha+1)}{\Gamma_q(m+1)} \frac{\theta(-acq)}{\theta(-c)}$$

where

$$\mathrm{d}\mu(x) = x^{\alpha} \mathrm{e}_q(-x) \mathrm{d}_q x$$

and

$$\mathrm{e}_q(x) = \sum_{n=0}^{\infty} \frac{X^n}{[n]!} = \frac{1}{(1-x)(1-xq)(1-xq^2)\cdots}$$

is the q-exponential function.

The previous results can again be restated as follows:

- The q-Laguerre polynomial $L_m^{(\alpha)}(x)$ is the eigenfunction with eigenvalue $[m]$ of

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- The operator $\mathcal{L}$ is self-adjoint with respect to the inner product

$$\langle f, g \rangle = \int_0^{c\cdot\infty} f(x)g(x)\mathrm{d}\mu(x)$$

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$$\langle 1, 1 \rangle = \int_0^\infty \mathrm{d}\mu(x) = \int_0^\infty x^\alpha e^{-x} \mathrm{d}x = \Gamma(\alpha + 1)$$

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Non-trivial example. The Lie algebra pair $(\mathrm{A}_{n-1}, \mathrm{A}_1)$

$$\int_{[0,\infty)^n} \prod_{i=1}^n x_i^\alpha \mathrm{e}^{-x_i} \prod_{1\le i<j\le n} |x_i - x_j|^{2\gamma}\, \mathrm{d}x_1\cdots\mathrm{d}x_n$$

$$= \prod_{i=0}^{n-1} \frac{\Gamma(\alpha+1+i\gamma)\Gamma(1+(i+1)\gamma)}{\Gamma(1+\gamma)}$$

For $x = (x_1, \ldots, x_n)$ let $\partial_{q,i}$ be the partial difference operator

$$(\partial_{q,i} f)(x) = \frac{f(x_1, \ldots, x_{i-1}, qx_i, x_{i+1}, \ldots, x_n) - f(x)}{(q-1)x_i}$$

Define the operators $E_r(q,t)$ acting on $\Lambda_n = \mathbb{Q}(q,t)[x_1, \ldots, x_n]^{\mathfrak{S}_n}$ by

$$E_r(q,t) = \sum_{i=1}^{n} x_i^r \left(\prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{tx_i - x_j}{x_i - x_j} \right) \partial_{q,i}$$

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Note that for $n = 1$

$$E_0(q,t) = \mathscr{D}_q \quad \text{and} \quad E_1(q,t) = x\,\mathscr{D}_q$$

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Lemma. For $r \geq 0$, $E_r : \Lambda_n \to \Lambda_n$

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition (a weakly decreasing sequence of nonnegative integers) and $\mathcal{L} : \Lambda_n \to \Lambda_n$ the operator

$$\mathcal{L} = E_1(q,t) + E_0(q,t) - q^{-\alpha-1} E_0(q^{-1}, t^{-1})$$

The eigenfunction of $\mathcal{L}$ with eigenvalue

$$t^{n-1}[\lambda_1] + \cdots + t[\lambda_{n-1}] + [\lambda_n]$$

defines the (q,t)-Laguerre polynomial $L_\lambda^{(\alpha)}(x) = L_\lambda^{(\alpha)}(x; q, t)$.

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The eigenfunction of $\mathcal{L}$ with eigenvalue

$$t^{n-1}[\lambda_1] + \cdots + t[\lambda_{n-1}] + [\lambda_n]$$

defines the (q,t)-Laguerre polynomial $L_\lambda^{(\alpha)}(x) = L_\lambda^{(\alpha)}(x; q, t)$.

Lemma. The $L_\lambda^{(\alpha)}(x)$ form a basis of Λ_n.

For $(q,t) = (q, q^{\gamma})$ with $q \to 1$ the (q,t)-Laguerre polynomials were studied by

Let $t = q^\gamma$ and define an inner product on Λ_n by

$$\langle f, g \rangle = \int_{x_1=0}^{c\cdot\infty} \int_{x_2=0}^{tx_1} \cdots \int_{x_n=0}^{tx_{n-1}} f(x)g(x)\mathrm{d}\mu(x)$$

where

$$\mathrm{d}\mu(x) = \prod_{1\le i<j\le n} x_i^{2\gamma}(q^{1-\gamma}x_j/x_i)_\gamma (x_j/x_i)_\gamma \prod_{i=1}^{n} x_i^\alpha \mathrm{e}_q(-x_i)\, \mathrm{d}_q x_i$$

and

$$(a)_z = \frac{(1-a)(1-aq)(1-aq^2)\cdots}{(1-aq^z)(1-aq^{z+1})(1-aq^{z+2})\cdots}$$

Theorem. The operator $\mathcal{L}$ is self-adjoint with respect to the inner product on Λ_n.

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Theorem c_λ can explicitly be computed in terms of "nice" special functions, such as theta functions and q-Gamma functions.

Ingredients in proofs are:

- Macdonald polynomials
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