A Discussion of Latin Interchanges

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Abstract

Latin interchanges have been used to establish the existence of critical sets in latin squares, to search for subsquare-free latin squares and to investigate the intersection sizes of latin squares. Donald Keedwell was the first to study latin interchanges in their own right. This paper builds on Keedwell's work, and proves new results about the existence of latin interchanges.

1 Definitions

A *latin square* $L = [l_{i,j}]$ of order n is an $n \times n$ array with entries chosen from a set $N = \{1, \dots, n\}$ in such a way that each element of N occurs precisely once in each row and column of the array. For example, the following array is a latin square of order 5:

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

For ease of exposition, a latin square will be represented by a set of ordered triples $\{(i, j; k) \mid$ where element k occurs in cell (i, j) of the array$\}$.

Two latin squares are said to be *isotopic* if one can be transformed into the other by rearranging rows, rearranging columns or renaming elements. Formally, let $L_1 = \{(i_1, j_1; k_1) \mid i_1, j_1, k_1 \in N\}$ and $L_2 = \{(i_2, j_2; k_2) \mid i_2, j_2, k_2 \in N\}$ be two latin squares of order n. Then L_1 is said to be isotopic to L_2 if there exist permutations α, β and γ such that $L_2 = \{(i_1\alpha, j_1\beta; k_1\gamma) \mid (i_1, j_1; k_1) \in L_1\}$. In this case L_2 is said to be an *isotope* of L_1. If $\alpha = \beta = \gamma$ then L_1 is said to be *isomorphic* to L_2.

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Each latin square $L = \{(i,j;k) \mid i,j,k \in N\}$ has five *conjugates* associated with it. These are:

- $L^* = \{(j,i;k) \mid (i,j;k) \in L\}$;
- ${}^{-1}L = \{(k,j;i) \mid (i,j;k) \in L\}$;
- $L^{-1} = \{(i,k;j) \mid (i,j;k) \in L\}$;
- ${}^{-1}(L^{-1}) = \{(j,k;i) \mid (i,j;k) \in L\}$; and
- $({}^{-1}L)^{-1} = \{(k,i;j) \mid (i,j;k) \in L\}$.

For more details on latin squares, isotopisms and conjugates, see [**8**].

A *partial latin square* P of order n is an $n \times n$ array with entries chosen from a set $N = \{1, \ldots, n\}$ in such a way that each element of N occurs at most once in each row and at most once in each column of the array.

Let I be a partial latin square of order n. Then $|I|$ is said to be the *size* of the partial latin square and the set of cells $\{(i,j) \mid (i,j;k) \in I, \text{ for some } k \in N\}$ is said to determine the *shape* of I. Let I and I' be two partial latin squares of the same order, with the same size and shape. Then I and I' are said to be *mutually balanced* if the entries in each row (and column) of I are the same as those in the corresponding row (and column) of I'. They are said to be *disjoint* if no cell in I' contains the same entry as the corresponding cell of I. A *latin interchange* I is a partial latin square for which there exists another partial latin square I', of the same order, size and shape with the property that I and I' are disjoint and mutually balanced. The partial latin square I' is said to be a *disjoint mate* of I.

Below is an example of a latin interchange of order 7 together with its disjoint mate. The size of the latin interchange is 9 as there are 9 non–empty cells.

$$\begin{array}{ccccccc}
\bullet & \bullet & \bullet & 3 & 4 & \bullet & 6 \\
\bullet & \bullet & 3 & 4 & \bullet & \bullet & \bullet \\
\bullet & \bullet & 4 & \bullet & 6 & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & 6 & \bullet & \bullet & \bullet & 3 \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet
\end{array}
\qquad
\begin{array}{ccccccc}
\bullet & \bullet & \bullet & 4 & 6 & \bullet & 3 \\
\bullet & \bullet & 4 & 3 & \bullet & \bullet & \bullet \\
\bullet & \bullet & 6 & \bullet & 4 & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & 3 & \bullet & \bullet & \bullet & 6 \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet
\end{array}$$

The concept of a latin interchange in a latin square is similar to the concept of a mutually balanced set (see [**19**]) or a trade (see [**15**]) in a block design.

Latin interchanges have been classified according to the number of elements appearing in the non–empty rows and columns. These configurations are referred to as *types* (see Keedwell [**17**]). (The term *type* will be rigorously defined in Section 3.) The type of the latin interchange above is

$$\begin{pmatrix} 2+2+2+3 \\ 2+2+2+3 \end{pmatrix},$$

as there are 3 entries in column 3 and 2 entries in each of columns 4, 5 and 7, and 3 entries in row 1 and 2 entries in each of rows 2, 3 and 5. Within each type one can further classify the latin interchanges according to their shape.

In this paper we discuss latin interchanges in detail, and build on the results obtained by Keedwell in [**17**]. Some general results pertaining to latin interchanges are stated, and we find all possible types of latin interchanges of size less than or equal to 11, and then all possible shapes for these types. We achieve these results by representing a latin interchange as a tripartite graph and then decomposing this graph into subgraphs.

2 History and Applications

A brief summary of the existing literature is given in this section. The motivation for studying latin interchanges comes from the fact that they are frequently used in the study of latin squares. To emphasise this point, three different problems which use the existence of latin interchanges in their solutions are documented in this section.

2.1 Subsquare-free latin squares

In [**11**] Elliot and Gibbons found subsquare-free latin squares of orders 16 and 18. A *subsquare* of order s is a latin square of order s obtained by deleting $n - s$ rows and columns from a latin square of order n. A *proper subsquare* has $1 < s < n$. A *subsquare-free latin square* is one which contains no proper subsquares.

Elliot and Gibbons used simulated annealing to generate subsquare free latin squares. One of the procedures they used involves the rearrangement of elements within a latin square. Thus a latin square is chosen and the elements are rearranged, two at a time, until all subsquares are eliminated. This procedure involves choosing two elements, x and y, from a row i, for some i. Assume x occurs in cell (i, j) and y occurs in (i, l). The entry x in cell (i, j) is replaced by y. The result is an array with two y's in column j, and no x's. To maintain the latin property, (that is, each element must occur precisely once in each row and column of the latin square), the original y (not the one in cell (i, j)) is replaced by an x. However, once this is done there are two x's in a row, so the original x (not the one in column j) is replaced by a y. This process is repeated until the y in cell (i, l) is replaced by an x. The result is a latin square distinct from the first, and hopefully one with fewer subsquares than the first. Using this rearranging process, Elliot and Gibbons were able to destroy all subsquares and obtain subsquare-free latin squares of orders 16 and 18.

This rearranging procedure is equivalent to identifying a latin interchange in which each column and each row contains precisely two non–empty cells. This latin interchange is then replaced by its disjoint mate. Elliot and Gibbons termed such latin interchanges *cycles*.

2.2 The intersection of two latin squares

Fu [**13**] investigated the problem of finding pairs of latin squares which intersect in a given number of elements. He defined the intersection of two latin squares as follows. Two latin squares $L = [l_{i,j}]$ and $M = [m_{i,j}]$ are said to have intersection k if there are exactly k cells (i, j) such that $l_{i,j} = m_{i,j}$. See also Fu, Fu and Guo [**12**] for results on the intersection of commutative latin squares, and Butler and Hoffman [**2**].

It is clear that if one removes from both L and M (where $L \neq M$) the cells which they have in common, then the remaining partial latin squares are examples of a latin interchange and its disjoint mate.

In [**13**] Fu verified that:

1. for any $n \geq 5$ there exists a pair of latin squares of order n which have intersection k, where $k \in \{0, 1, 2, \ldots, n^2 - 6, n^2 - 4, n^2\}$;

2. for $n = 1, 2, 3, 4$ there exists a pair latin squares of order n which have intersection $k \in \{0, 1\}$, $k \in \{0, 4\}$, $k \in \{0, 3, 9\}$ and $k \in \{0, 1, 2, 3, 4, 6, 8, 9, 12, 16\}$ respectively.

Therefore his results verify that:

1. for any $n \geq 5$ there exists a partial latin square of order n which has the properties of a latin interchange and is of size h, where $h \in \{4, 6, \ldots, n^2 - 2, n^2 - 1, n^2\}$;

2. for $n = 2, 3, 4$ there exists a partial latin square of order n which has the properties of a latin interchange and is of size h, where $h \in \{4\}$, $h \in \{6, 9\}$ and $h \in \{4, 7, 8, \ldots, 14, 15, 16\}$ respectively.

2.3 Critical sets

The papers [**3**], [**4**], [**6**], [**7**], [**9**], [**10**], [**17**] and [**23**] address the problem of finding critical sets for latin squares. A *critical set* is a partial latin square C of order n, with the property that it is contained in a unique latin square L of order n, and such that any subset of C is contained in at least two latin squares of order n. When proving that a partial latin square C is a critical set, one must verify that the partial latin square is contained in precisely one latin square of order n and that for each subset of C there are at least two latin squares of order n which intersect in this subset. In other words, one needs to show that for each entry in the critical set there exists a latin interchange in L which intersects the critical set in this element alone. Therefore establishing the existence of latin interchanges in given latin squares is a vital part of finding critical sets for latin squares.

2.4 Latin interchanges in their own right

Even though latin interchanges have been used extensively in the study of latin squares, very few papers have appeared which deal specifically with the properties of latin interchanges. To the authors' knowledge there are only two such papers, [**17**] and [**5**].

In 1993 Keedwell, [**17**], rigorously defined a latin interchange, though it should be noted that Keedwell used the term *critical partial latin square* for a latin interchange. In [**17**], Keedwell categorised latin interchanges by their types, then proceeded to show that latin interchanges of every possible type exist for all sizes up to 10 inclusive. In this paper we extend these results to 11.

In 1994 Cooper and Donovan [**5**] presented a paper which discussed the possible representations for a latin interchange. They pointed out that a latin interchange may be thought of as a decomposition of a tripartite graph into triangles. This representation forms the basis for the results presented in this paper.

2.5 The paper at hand

This paper extends Keedwell's results and verifies that latin interchanges of every possible type exist for all sizes up to 11 inclusive. A review of some basic results obtained by Keedwell will be given in Section 3. Additionally, this section also includes some basic techniques for constructing latin interchanges.

In his paper [**17**], Keedwell listed the permissible types of latin interchanges of sizes up to 10. Once the possible types have been established, this information can then be used to find the possible shapes for latin interchanges of a given size. Keedwell found latin interchanges of all possible shapes for sizes 4, 5, 6, 7, and 8. In the case of latin interchanges of size 9 he listed four different shapes, and in the case of size 10 he listed 22 different shapes. In this paper we extend Keedwell's results and find all possible shapes for size 9, of which there are seven (it should be noted that in this case the three new types are conjugates of ones given by Keedwell), size 10, of which there are 40, and size 11, of which there are 62 possible shapes. The possible shapes of latin interchanges of size 10, not documented by Keedwell, and size 11 are listed in Section 5.

In searching for latin interchanges of size 11 we first generated all possible partial latin squares of order 5 with precisely 11 non–empty cells, then identified those which satisfy the necessary conditions for a latin interchange. These are mentioned in the following section. Due to the large number of possible latin interchanges, it was necessary to computerise the search and develop efficient algorithms to implement the search. We began by compiling a list of all possible partial latin squares. To achieve this we noted that a partial latin square may be thought of as a decomposition of a tripartite graph into triangles. We then generated all possible tripartite graphs with a given set of parameters corresponding to each type. These were then checked to see which ones could be decomposed into triangles. At this point we had a list of all relevant partial latin squares, and the next step was to efficiently identify which of these were latin interchanges. This was done by modifying the tripartite graphs and then checking to see if the modified graph could be decomposed into copies of K_4 minus an edge. The representation of a latin interchange as a decomposition of a tripartite graph into copies of K_4 minus an edge is the central idea in this paper and the basis for our searching technique. The justification for this representation of a latin interchange is given in Section 4.

3 Keedwell's paper and more

Keedwell's paper [**17**] was the first to study latin interchanges in detail and so the following definitions and background information have been taken from this paper.

Let I be a latin interchange of order n. Let r denote the number of non–empty rows, c the number of non–empty columns and e the number of elements of N which occur at least once in I. We will use r_i to denote the number of non–empty cells in row i of the latin interchange I, c_j to denote the number of non–empty cells in column j of the latin interchange I, and e_k to denote the number of times the element k occurs in the latin interchange I, for $i, j, k = 1, \ldots, n$. It is obvious that

$$\sum_{i=1}^{n} r_i = \sum_{j=1}^{n} c_j = \sum_{k=1}^{n} e_k = |I|.$$

Thus the rows, (columns, elements), partition the elements of the latin interchange I, and this partition must satisfy the conditions given by Gale [**14**] and Ryser [**20**] or pages 61–65 of [**21**]. Keedwell used these partitions to define the *type* of a latin interchange. So the *type* of the latin interchange I is

$$\left(\begin{array}{c} c_1 + c_2 + c_3 + \ldots + c_n \\ r_1 + r_2 + r_3 + \ldots + r_n \end{array} \right).$$

Note that the type describes the number of non–empty cells in the columns and rows of I. Since the empty rows and columns of I give very little useful information, where ever possible they are deleted, and the latin interchange I is taken to be a partial latin square of order n, where $n = max\{r, c\}$. Then the type is written as

$$\left(\begin{array}{c} c_1 + c_2 + c_3 + \ldots + c_c \\ r_1 + r_2 + r_3 + \ldots + r_r \end{array} \right).$$

Any isotope of a latin interchange is a latin interchange. Therefore it will be assumed that $0 < c_1 \leq c_2 \leq c_3 \leq \ldots \leq c_c \leq n$ and $0 < r_1 \leq r_2 \leq r_3 \leq \ldots \leq r_r \leq n$, and any two latin interchanges which are isotopic are said to be of the same type. Similarly it follows that the conjugate of a latin interchange is a latin interchange. However, since the definition of type only refers to the non–empty columns and rows of I, the following distinctions are made. Two latin interchanges which are the transpose of one another are said to be of the same type. The remaining conjugates will be listed separately.

The following constraints on r_i, c_j and e_k may be deduced.

LEMMA 3.1 *Let I be a latin interchange of order n. Then for all $1 \leq i, j, k \leq n$,*

$$r_i \geq 2, \quad c_j \geq 2 \quad \text{and} \quad e_k \geq 2.$$

Proof. If I is a latin interchange, then by definition there exists a partial latin square I' of the same shape and size, which is disjoint from I and mutually balanced. It follows that these inequalities hold.

□

COROLLARY 3.2 *If I is a latin interchange of order n, then for $1 \leq i, j, k \leq n$,*

$$\begin{aligned} r_i &\leq |I|/2, \\ c_j &\leq |I|/2 \\ \text{and} \quad e_k &\leq |I|/2. \end{aligned}$$

COROLLARY 3.3 *If I is a latin interchange with $r = 2$, then $r_1 = r_2$. Similarly, if $c = 2$ then $c_1 = c_2$, and if $e = 2$ then $e_1 = e_2$.*

COROLLARY 3.4 *A latin interchange of size 5 cannot exist.*

LEMMA 3.5 *Let I be a latin interchange of order n. Then*

$$\begin{aligned} \forall i \quad 1 \leq i \leq n, &\quad c \geq r_i; \\ \forall k \quad 1 \leq k \leq n, &\quad c \geq e_k; \\ \forall j \quad 1 \leq j \leq n, &\quad r \geq c_j; \\ \forall k \quad 1 \leq k \leq n, &\quad r \geq e_k; \\ \forall i \quad 1 \leq i \leq n, &\quad e \geq r_i; \\ \forall j \quad 1 \leq j \leq n, &\quad e \geq c_j. \end{aligned}$$

We now list some methods which can be used to construct latin interchanges. The first such method is an obvious construction, but has been included for completeness.

LEMMA 3.6 *Let $P = \{(p_i, p_j; p_k) \mid p_i, p_j, p_k \in N\}$ and $Q = \{(q_i, q_j; q_k) \mid q_i, q_j, q_k \in N\}$ be latin interchanges of orders p and q respectively. Assume that P has type*

$$\begin{pmatrix} c_{p1} + c_{p2} + c_{p3} + \ldots + c_{pc} \\ r_{p1} + r_{p2} + r_{p3} + \ldots + r_{pr} \end{pmatrix},$$

and Q has type

$$\begin{pmatrix} c_{q1} + c_{q2} + c_{q3} + \ldots + c_{qc} \\ r_{q1} + r_{q2} + r_{q3} + \ldots + r_{qr} \end{pmatrix}.$$

Then there exists a latin interchange $P + Q$ of order $p + q$ and type

$$\begin{pmatrix} c_{p1} + c_{p2} + c_{p3} + \ldots + c_{pc} + c_{q1} + c_{q2} + c_{q3} + \ldots + c_{qc} \\ r_{p1} + r_{p2} + r_{p3} + \ldots + r_{pr} + r_{q1} + r_{q2} + r_{q3} + \ldots + r_{qr} \end{pmatrix}.$$

Proof. Let $P+Q$ be the partial latin square

$$\{(p_i,p_j;p_k),(p+q_i,p+q_j;p+q_k) \mid (p_i,p_j;p_k)\in P \wedge (q_i,q_j;q_k)\in Q\}.$$

To show that $P+Q$ is a latin interchange we must find a partial latin square which is the disjoint mate of $P+Q$. Consider the partial latin square

$$P'+Q' = \{(p_i,p_j;p'_k),(p+q_i,p+q_j;p+q'_k) \mid (p_i,p_j;p'_k)\in P' \wedge (q_i,q_j;q'_k)\in Q'\},$$

where P' and Q' are the disjoint mates of P and Q respectively. The result is immediate.

□

The latin interchanges of size 8, types 1a, 1b, 1c, and size 10, types 1a, 1b, 1c given by Keedwell [**17**] can be constructed using this result or a variation of it where the rows, columns or elements have been appropriately relabelled.

One may vary this method by placing the latin interchanges one on top of the other or side by side. For example, assume $pc \geq qc$, then P and Q may be used to construct a latin interchange of type

$$\begin{pmatrix} (c_{p1}+c_{q1})+(c_{p2}+c_{q2})+(c_{p3}+c_{q3})+\ldots+(c_{qc}+c_{qc})+\ldots+c_{pc} \\ r_{p1}+r_{p2}+r_{p3}\ldots+r_{pr}+r_{q1}+r_{q2}+r_{q3}+\ldots+r_{qr} \end{pmatrix}.$$

In Keedwell's paper the latin interchanges of size 8, types 2, 4a, and size 10 types 2, 4, 5a, 7a, 8a, 11a, have been constructed using this result or a variation if it where the rows, columns or elements have been appropriately relabelled.

The next method of construction is analogous to the construction of group divisible designs given by Hanani in [**16**].

LEMMA 3.7 *Let $P=\{(p_i,p_j;p_k) \mid p_i,p_j,p_k\in N\}$ and $Q=\{(q_i,q_j;q_k) \mid q_i,q_j,q_k\in N\}$ be latin interchanges of order p and q respectively. Assume that P has type*

$$\begin{pmatrix} c_{p1}+c_{p2}+c_{p3}+\ldots+c_{pc} \\ r_{p1}+r_{p2}+r_{p3}+\ldots+r_{pr} \end{pmatrix},$$

and Q has type

$$\begin{pmatrix} c_{q1}+c_{q2}+c_{q3}+\ldots+c_{qc} \\ r_{q1}+r_{q2}+r_{q3}+\ldots+r_{qr} \end{pmatrix}.$$

Then there exists a latin interchange PQ of order pq and type

$$\begin{pmatrix} c_{p1}c_{q1}+\ldots+c_{p1}c_{qr}+c_{p2}c_{q1}+\ldots+c_{p2}c_{qr}+\ldots\ldots+c_{pc}c_{q1}+\ldots+c_{pc}c_{qr} \\ r_{p1}r_{q1}+\ldots+r_{p1}r_{qr}+r_{p2}r_{q1}+\ldots+r_{p2}r_{qr}+\ldots\ldots+r_{pc}r_{q1}+\ldots+r_{pc}r_{qr} \end{pmatrix}.$$

Proof. In this case we simply take the direct product of the latin interchange P with the latin interchange Q. Thus PQ is taken to be a $pq \times pq$ array which contains entries as follows:

$$PQ = \{((p_i, q_i), (p_j, q_j); (p_k, q_k)) \mid (p_i, p_j; p_k) \in P \wedge (q_i, q_j; q_k) \in Q\}$$

To show that PQ is a latin interchange we need to establish that it has a disjoint mate. Since P and Q are latin interchanges there exists partial latin squares P' and Q' which are the disjoint mates of P and Q respectively. Let

$$PQ' = \{((p_i, q_i), (p_j, q_j); (p'_k, q'_k) \mid (p_i, p_j; p'_k) \in P' \wedge (q_i, q_j; q'_k) \in Q'\}.$$

Is is easy to check that PQ' has the same shape and size as PQ, that it is disjoint from PQ and that PQ' and PQ are mutually balanced. Therefore the result follows.

□

We close this section with two constructions which increase the size of the latin interchange by 2 and by 3. These two constructions are illustrated with the following examples. The first example shows how a latin interchange of size 4 (given on the left) can be extended to a latin interchange of size 6 (given on the right). The next example illustrates how this same latin interchange can be extended to a latin interchange of size 7 (once again given on the right).

$$\begin{array}{cc} 1 & 2 \\ 2 & 1 \\ & \end{array} \qquad\qquad \begin{array}{ccc} 1 & 2 & 3 \\ 2 & 3 & 1 \\ \bullet & \bullet & \bullet \end{array}$$

$$\begin{array}{cc} 1 & 2 \\ 2 & 1 \\ & \end{array} \qquad\qquad \begin{array}{ccc} 1 & 2 & 3 \\ 2 & 1 & \bullet \\ \bullet & 3 & 1 \end{array}$$

These two methods of construction are formalised in the following lemma.

LEMMA 3.8 *Let* $P = \{(p_i, p_j; p_k) \mid p_i, p_j, p_k \in N\}$ *be a latin interchange of order* p. *Let* P *have type*

$$\left(\begin{array}{c} c_{p1} + c_{p2} + c_{p3} + \ldots + c_{pc} \\ r_{p1} + r_{p2} + r_{p3} + \ldots + r_{pr} \end{array} \right).$$

Assume that for some $p_z \in N$

$$|\{(p_i, p_j; p_z) \mid (p_i, p_j; p_z) \in P\}| = 2,$$

and that element p_z *occurs in cells* (p_s, p_u) *and* (p_t, p_v) *of* P. *For ease of exposition assume that the number of non–empty cells in columns* p_u *and* p_v *is* c_{pu} *and* c_{pv} *respectively, and that the number of non–empty cells in rows* p_s *and* p_t *is* r_{ps} *and* r_{pt} *respectively. Then*

1. *there exists a latin interchange* P_2 *of order* $p+1$ *and type*

$$\left(\begin{array}{c} 2 + c_{p1} + c_{p2} + c_{p3} + \ldots + c_{pc} \\ r_{p1} + r_{p2} + \ldots + r_{ps-1} + (r_{ps}+1) + r_{ps+1} + \ldots + r_{pt-1} + (r_{pt}+1) \\ \qquad\qquad\qquad\qquad + r_{pt+1} + \ldots + r_{pr} \end{array} \right),$$

2. *there exists a latin interchange* P_3 *of order* $p+1$ *and type*

$$\left(\begin{array}{c} 2 + c_{p1} + c_{p2} + c_{p3} + \ldots + c_{pv-1} + (c_{pv}+1) + c_{pv+1} + \ldots + c_{pc} \\ 2 + r_{p1} + r_{p2} + r_{p3} + \ldots + r_{ps-1} + (r_{ps}+1) + r_{ps+1} + \ldots + r_{pr} \end{array} \right).$$

Proof.

1. Take a $p+1 \times p+1$ array and place the following entries in this array.

   - $(p_i, p_j; p_k)$ where $(p_i, p_j; p_k) \in P \setminus \{(p_s, p_u; p_z)\}$
   - $(p_s, p_u; \infty), (p_s, p+1; p_z), (p_t, p+1; \infty)$

   To show that this partial latin square is a latin interchange we need to find its disjoint mate. Take P_2' to be a $p+1 \times p+1$ array which contains the following entries:

   - $(p_i, p_j; p_k')$ where $(p_i, p_j; p_k) \in P' \setminus \{(p_t, p_u; p_z)\}$
   - $(p_t, p_u; \infty), (p_s, p+1; \infty), (p_t, p+1; p_z)$,

   where P' is the disjoint mate of P. It is easy to see that P_2' has the same shape and size as P_2. It will certainly be disjoint and mutually balanced in all rows other than p_s and p_t and all columns other than p_u, p_v and $p+1$. In P_2 we have added an ∞ to rows p_s and p_t, but otherwise kept them the same as P. Since ∞ occurs in rows p_s and p_t of P_2', P_2 and P_2' are mutually balanced in these rows. Since p_z only occurred in cells (p_s, p_u) and (p_t, p_v) of P, p_z must occur in cells (p_s, p_v) and (p_t, p_u) of P'. We have taken p_z out of column p_u of P_2 and replaced it by ∞. However we have done the same thing to column p_u in P_2' and so they are mutually balanced in column p_u. We have not changed column p_v in either partial latin square so they are mutually balanced in this column. Finally we have added p_z and ∞ to column $p+1$ in both P_2 and P_2' so they are mutually balanced in this column and therefore mutually balanced over all.

   The result now follows.

2. Take P_3 to be the $p+1 \times p+1$ array containing the following entries

   - $(p_i, p_j; p_k)$ where $(p_i, p_j; p_k) \in P$ and
   - $(p_s, p+1; \infty), (p+1, p_v; \infty), (p+1, p+1; p_z)$.

   To show that P_3 is a latin interchange we need to find its disjoint mate. Take P_3' to be a $p+1 \times p+1$ array which contains the following entries:

- $(p_i, p_j; p'_k)$ where $(p_i, p_j; p'_k) \in P' \setminus \{(p_s, p_v; p_z)\}$
- $(p_s, p_v; \infty), (p_s, p+1; p_z), (p+1, p_v; p_z), (p+1, p+1; \infty)$

where P' is the disjoint mate of P. It is easy to see that P'_3 is the same shape and size as P_3. It will certainly be disjoint and mutually balanced in all rows other than p_s and $p+1$ and all columns other than p_v and $p+1$. We have added an ∞ to row p_s, and ∞ and p_z to row $p+1$ of P and P' to obtain P_3 and P'_3 resectively. Therefore P_3 and P'_3 are mutually balanced in these rows. A similar argument verifies that they are mutually balanced in all columns and so they are mutually balanced.

The result now follows.

□

This method may be used to construct the latin interchanges of size 6, type 2, size 7, type 1, size 8, type 4b, 5, size 9, type 2 and size 10, types 4, 6, 7a, 7b, 8b, 9, 10a, 10b, 11b and 12 given by Keedwell [**17**].

4 Graph representation

A latin square L of order n corresponds to a decomposition of the complete tripartite graph $K_{n,n,n}$ into triangles. Such a representation may be achieved as follows. For each $i, j, k \in \{1, \dots, n\}$ assign row i the vertex v_i, column j the vertex v_{n+j} and element k the vertex v_{2n+k}. For each triple $(i, j; k)$ of the latin square L take an edge from vertex v_i to v_{n+j}, another from vertex v_i to v_{2n+k} and finally one from vertex v_{n+j} to v_{2n+k}. For each cell (i, j) of L there is precisely one element $k \in N$ such that k occurs in cell (i, j) of L. Hence for each pair of vertices v_i, v_{n+j}, $1 \le i, j \le n$ there is one edge from v_i to v_{n+j}. Since L is a latin square each element of N occurs once in each row. Therefore, for each pair of vertices v_i, v_{2n+k}, $1 \le i, k \le n$ there is one edge from v_i to v_{2n+k}. The same holds for each column of L, and so for each pair of vertices v_{n+j}, v_{2n+k}, $1 \le n, k \le n$ there is one edge from v_{n+j} to v_{2n+k}. Now the edges (v_i, v_{n+j}), (v_i, v_{2n+k}) and (v_{n+j}, v_{2n+k}) form a triangle. Using a similar argument one can show that any decomposition of the complete tripartite graph $K_{n,n,n}$ into triangles can be represented as a latin square of order n.

Consequently a partial latin square of order n may be represented as a subgraph of the complete tripartite graph $K_{n,n,n}$. Further, this subgraph can be decomposed into triangles. However not all partial latin squares are latin interchanges so we need to extend these ideas to distinguish latin interchanges from partial latin squares.

In what follows, the terminology $K_4 \setminus \{e\}$ is used to represent K_4 with one edge removed.

Construction 1:

Let P be a partial latin square of order n, and G a multigraph constructed as follows. Let the vertex set of G be $\{v_1, \dots, v_n\} \cup \{v_{n+1}, \dots, v_{2n}\} \cup \{v_{2n+1}, \dots, v_{3n}\}$.

For each $i, j, k \in \{1, 2, \ldots, n\}$ assign row i of P the vertex v_i, column j the vertex v_{n+j} and element k the vertex v_{2n+k}. For each triple $(i, j; k)$ of P take one edge from vertex v_i to v_{n+j}, two edges from vertex v_i to v_{2n+k} and two edges from vertex v_{n+j} to v_{2n+k}.

The following lemma verifies that if P is a latin interchange, then the graph constructed in this manner can be decomposed into copies of $K_4 \setminus \{e\}$.

LEMMA 4.9 *Let P be a partial latin square of order n and G a multigraph constructed as stated above. If P is a latin interchange, then the multigraph G can be decomposed into copies of $K_4 \setminus \{e\}$.*

Proof. Assume that P is a latin interchange, and so P is partial latin square of order n, as is its disjoint mate P'.

Construct a multigraph G' with vertices $v_1, \ldots, v_{3n}$ and edges as follows.

1. Draw an edge between the vertices v_x and v_{n+y}, v_x and v_{2n+z} and v_{n+y} and v_{2n+z}, for $x, y, z \in \{1, \ldots, n\}$, if and only if element z occurs in cell (x, y) of P.

2. Draw an edge between the vertices v_x and v_{2n+z} and between v_{n+y} and v_{2n+z}, for $x, y, z \in \{1, \ldots, n\}$, if and only if element z occurs in cell (x, y) of P'.

Since P' is the disjoint mate of P, it has the same size and shape as P, and the two partial latin squares are mutually balanced. Therefore, if in Step 2 an edge is drawn from vertex v_x to v_{2n+z}, for some x and z, there must have been an edge drawn from v_x to v_{2n+z} in Step 1. Likewise, for any edge from v_{n+y} to v_{2n+z}. By the same reasoning any edge drawn from v_x to v_{2n+z} or v_{n+y} to v_{2n+z} in Step 1 must have a corresponding edge drawn in Step 2. Hence the graphs G and G' are identical.

Since P is a partial latin square the edges obtained from Step 1 can be decomposed into triangles, where the triangles correspond to a triple $(x, y; z)$ of P. Each such triangle contains an edge from vertex v_x to vertex v_{n+y}, for some $x, y \in \{1, \ldots, n\}$. This edge corresponds to the cell (x, y) of P and the third vertex of this triangle corresponds to an element z which occurs in this cell of P. Select one triple, say $(i, j; k)$ of P. The disjoint mate P' of P has element k', $k' \neq k$, in the cell (i, j). So using Step 2, G must also contain edges from the vertices v_i to $v_{2n+k'}$ and from v_{n+j} to $v_{2n+k'}$. These two edges combined with the edges from v_i to v_{n+j}, v_i to v_{2n+k} and v_{n+j} to v_{2n+k} form a $K_4 \setminus \{e\}$. As the triple $(i, j; k)$ runs over all the triples of P we obtain a set of copies of $K_4 \setminus \{e\}$. Since P' is a partial latin square and has the same shape and size as P, these copies of $K_4 \setminus \{e\}$ form a decomposition of G.

□

The next theorem demonstrates that this process can be reversed and used to check whether or not a partial latin square is a latin interchange.

THEOREM 4.10 *Let P be a partial latin square of order n and G a multigraph constructed as in Construction 1. Suppose that the multigraph G can be decomposed into copies of $K_4 \setminus \{e\}$ in such a way that for each $K_4 \setminus \{e\}$ on the vertices v_i, v_{n+j}, v_{2n+k} and $v_{2n+k'}$, $1, \leq i, j, k \leq n$, there exists a unique element $(i, j; k)$ in P. Then P is a latin interchange.*

Proof. Assume that P is a partial latin square with multigraph G which can be decomposed into copies of $K_4 \setminus \{e\}$ satisfying the condition as stated. (This condition ensures that the copies of $K_4 \setminus \{e\}$ cover the triangles which correspond to the elements of P. For more details see below.) We begin by showing that each $K_4 \setminus \{e\}$ has one vertex from the set $V_i = \{v_1, \ldots, v_n\}$, one vertex from the set $V_j = \{v_{n+1}, \ldots, v_{2n}\}$ and two vertices from the set $V_k = \{v_{2n+1}, \ldots, v_{3n}\}$.

A counting argument verifies that G must contain $|P|$ copies of $K_4 \setminus \{e\}$.

Since G is a tripartite graph and since a $K_4 \setminus \{e\}$ has four vertices and five edges, two of these vertices must be from the same set V_i, V_j or V_k, and four edges must have an endpoint in this set. Take one $K_4 \setminus \{e\}$ and label it K. Assume that K has two vertices from the set $\{v_1, \ldots, v_n\}$. Label these vertices v_x and $v_{x'}$. Label the remaining vertices of K, v_{n+y} and v_{2n+z}. It follows that K must contain the edges (v_x, v_{n+y}) and $(v_{x'}, v_{n+y})$. Since G contains $|P|$ edges from the vertices $v_1, \ldots, v_n$, to $v_{n+1}, \ldots, v_{2n}$, there remains $|P| - 2$ such edges to be used in the remaining copies of $K_4 \setminus \{e\}$. There are a further $|P| - 1$ copies of $K_4 \setminus \{e\}$ in the decomposition, and so there exists at least one $K_4 \setminus \{e\}$ which does not have an edge from the set of vertices $\{v_1, \ldots, v_n\}$ to the set of vertices $\{v_{n+1}, \ldots, v_{2n}\}$. This leads to a contradiction. Similarly it can be shown that K cannot contain two vertices from the set $\{v_{n+1}, \ldots, v_{2n}\}$. Hence each $K_4 \setminus \{e\}$ must be on vertices of the form v_x, v_{n+y}, v_{2n+z} and $v_{2n+z'}$, where $x, y, z, z' \in \{1, 2, \ldots, n\}$ and edges of the form (v_x, v_{n+y}), (v_x, v_{2n+z}), $(v_x, v_{2n+z'})$, (v_{n+y}, v_{2n+z}) and $(v_{n+y}, v_{2n+z'})$.

Each $(i, j; k)$ of P corresponds to a different $K_4 \setminus \{e\}$ in the decomposition of G. Fix $(i, j; k)$ of P and let K be the corresponding $K_4 \setminus \{e\}$. Let the vertices of K be v_i, v_{n+j}, v_{2n+k} and $v_{2n+k'}$ for some $v_{2n+k'} \in V_k$. Now place $(i, j; k')$ in P'. Repeat this process for each $(i, j; k) \in P$.

If it can be shown that P' is a disjoint mate of P then it follows that P is a latin interchange.

It is obvious that P' has the same shape and size as P. Since v_{2n+k} and $v_{2n+k'}$ are distinct vertices of G, it follows that P and P' are disjoint. The edges $(v_i, v_{2n+k'})$ and $(v_{n+j}, v_{2n+k'})$ of K indicate that the element k' occurred in row i and column j of P and so it follows that P and P' are mutually balanced.

Hence P is a latin interchange.

□

It should be noted that the multigraph G constructed using Construction 1 is the same for both of the partial latin squares given below. However the first is not a latin interchange. These two examples show that extra condition placed on the decomposition of G is necessary.

1	2	•		1	2	•
2	1	3		2	3	1
•	3	1		•	1	3

This last result has been used to develop algorithms to search for all possible latin interchanges of size 11 and less. The results of this search are given in Section 5.

5 Latin interchanges of size 11 or less

In this section we outline the algorithm used to search for non–isomorphic latin interchanges of size up to and including 11. The main steps of the algorithm are as follows.

First of all determine the types of latin interchanges that are to be searched for. Each type is represented by a tripartite graph and, if possible, this is decomposed into triangles. The decomposition is performed by the program *autogen*, a graph decomposition program developed by Peter Adams (see [1]).

If a decomposition is possible then the original tripartite graph represents a partial latin square. In this case the original tripartite graph is modified as in Construction 1 and returned to *autogen* which attempts to decompose it into copies of $K_4 \setminus \{e\}$. If this is possible then a latin interchange has been found. A more detailed discussion of the algorithm is given below.

5.1 Finding latin interchanges of size m using *autogen*

In the following, r denotes the number of rows in a type, c denotes the number of columns and e denotes the number of elements.

1. For a given value m, find all the types

$$\begin{pmatrix} c_1 + c_2 + c_3 + \ldots + c_n \\ r_1 + r_2 + r_3 + \ldots + r_n \end{pmatrix}$$

   where $\sum_{j=1}^{n} c_j = \sum_{i=1}^{n} r_i = m$ and such that the values of c_j and r_i satisfy the requirements of Lemmas 3.1, and Corollaries 3.2 and 3.3. These are then labelled as Types $1, 2, 3 \ldots$.

   Observe that

$$\begin{pmatrix} c_1 + c_2 + c_3 + \ldots + c_n \\ r_1 + r_2 + r_3 + \ldots + r_n \end{pmatrix} \text{and} \begin{pmatrix} r_1 + r_2 + r_3 + \ldots + r_n \\ c_1 + c_2 + c_3 + \ldots + c_n \end{pmatrix}$$

   are transposes of each other.

   If $\max\{c_j\} > \max\{r_i\}$, then for $1 \leq i \leq r$ and $1 \leq j \leq c$, relabel the r_i's as c_j's. That is, consider the transpose.

   Split each type into subtypes according to the number of distinct elements e that may be used where $\max\{r_i : 1 \leq i \leq r\} \leq e \leq \lfloor \frac{m}{2} \rfloor$. One subtype is created for each value of e in this range. These are then labelled as Subtypes $1a, 1b, 1c \ldots$

2. For each subtype we have a value for r, c and e. Convert each subtype into a tripartite graph with r vertices in the first partition, c in the second and e in the third. Let vertex v_0 denote row r_r, for this row has the maximum number of elements occurring in it, and hence v_0 is a vertex of maximum degree in the graph. Let the remaining $r-1$ rows be denoted by vertices $v_1, v_2, \ldots, v_{r-1}$, noting that the degree of vertex v_x is the value r_x of the row that it denotes. Similarly, let vertex v_{r+j} for $0 \leq j \leq c-1$ denote columns $c_1, c_2, \ldots, c_c$ and let vertex v_{r+c+k} for $0 \leq k \leq e-1$ denote the elements $e_1, e_2, \ldots, e_e$. Without loss of generality, r_r triangles of the tripartite graph can be fixed. These are (v_0, v_i, v_{i+c}) for $r \leq i \leq r+e-1$.

3. Use the parameters determined in Step 2 as input to *autogen*, and attempt to decompose the correponding graph of each subtype into triangles.

4. Classify any solutions that are found into isomorphism classes. This can be done, for example, by using the program *nauty* see B McKay [18].

5. Take a representative set of m triangles from each isomorphism class and modify the underlying graph as follows. For each triangle $(v_i, v_j; v_k)$ take one edge from vertex v_i to v_{r+j}, two edges from vertex v_i to v_{r+c+k} and two edges from vertex v_{r+j} to v_{r+c+k}.

6. Return the tripartite graph of Step 5 to *autogen* and decompose it into copies of $K_4 \setminus \{e\}$ satisfying the following condition. Recall that one vertex of each $K_4 \setminus \{e\}$ represents a row, another represents a column and the last two represent two distinct elements. We require that each copy of a $K_4 \setminus \{e\}$ corresponds to a unique triangle from the first decomposition.

If a solution is found in Step 6 then a latin interchange of this subtype exists.

The following table lists the number of non–isomorphic latin interchanges that exist for all sizes less than 12.

m	4	5	6	7	8	9	10	11
Number of latin interchanges	1 (1)	0 (0)	2 (2)	1 (1)	9 (9)	7 (4)	40 (22)	62 –

The number of latin interchanges of sizes less than 12.
Figures in brackets represent the numbers previously found by Keedwell.

Table 1 lists all the latin interchanges of size 10 that were not found by Keedwell, and Table 2 lists the latin interchanges of size 11.

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Type

$$\begin{pmatrix} 2+2+2+2+2 \\ 2+2+3+3 \end{pmatrix}$$

Possible non–isomorphic shapes

1 2 3 ● ● 2 3 1 ● ● ● ● ● 1 2 ● ● ● 2 1	1 2 3 ● ● 2 3 1 ● ● ● ● ● 1 4 ● ● ● 4 1	1 2 3 ● ● 2 3 1 ● ● ● ● ● 4 5 ● ● ● 5 4	1 2 3 ● ● 3 1 ● 2 ● ● ● 2 ● 3 ● ● ● 3 2

Type

$$\begin{pmatrix} 2+2+2+4 \\ 2+2+3+3 \end{pmatrix}$$

Possible non–isomorphic shapes

1 2 3 ● 2 3 1 ● 3 ● ● 4 4 ● ● 3	1 2 5 ● 2 5 1 ● 3 ● ● 4 4 ● ● 3	1 2 3 ● 2 3 ● 4 3 ● 1 ● 4 ● ● 3	1 2 3 ● 2 1 ● 4 3 ● 2 ● 4 ● ● 1

Type

$$\begin{pmatrix} 2+2+3+3 \\ 2+2+3+3 \end{pmatrix}$$

Possible non–isomorphic shapes

1 2 3 ● 2 3 ● 1 ● ● 1 2 ● ● 2 3	1 2 3 ● 2 3 ● 1 ● ● 1 4 ● ● 4 3	1 2 3 ● 2 3 ● 1 ● 1 ● 2 3 ● 1 ●	1 2 3 ● 2 3 ● 4 ● 4 ● 2 3 ● 1 ●
1 2 3 ● 2 1 ● 3 ● 3 ● 1 3 ● 1 ●	1 2 3 ● 2 4 ● 3 ● 3 ● 4 3 ● 1 ●	1 2 3 ● 2 1 ● 4 ● 4 ● 2 3 ● 2 ●	

Type

$$\begin{pmatrix} 2+2+3+3 \\ 3+3+4 \end{pmatrix}$$

Possible non–isomorphic shapes

	1 2 3 4 3 1 4 ● 2 4 ● 3	1 2 3 4 3 4 1 ● 4 1 ● 2	

Type

$$\begin{pmatrix} 2+2+3+3 \\ 2+4+4 \end{pmatrix}$$

Possible non–isomorphic shapes

	1 2 3 4 4 3 1 2 3 4 ● ●	

Table 1: New latin interchanges of size 10

Type $\begin{pmatrix} 2+2+2+2+3 \\ 2+2+2+2+3 \end{pmatrix}$ Possible non–isomorphic shapes					
1 2 3 • 2 3 • • 3 • • 1 • • • 1 3 • • • 3 1	1 2 3 • • 2 1 • • • 3 • 2 • • • • • 2 3 • • • 3 2	1 2 3 • • 2 1 • • • 3 • 2 • • • • • 1 2 • • • 2 1	1 2 3 • • 2 • • 1 • • 3 • 2 • • • 1 • 3 • • • 3 1	1 2 3 • • 2 1 • • • 3 • 2 • • • • • 1 3 • • • 3 1	1 2 3 • • 2 1 • • • 3 • • 2 • • • 2 • 3 • • • 3 2
1 2 3 • • 2 • • 1 • 3 • • • 2 • 1 • 2 • • • 2 • 3	1 2 3 • • 2 1 • • • • 3 1 • • • • • 1 4 • • • 4 1	1 2 3 • • 2 1 • • • • 3 1 • • • • • 3 4 • • • 4 3	1 2 3 • • 2 1 • • • • 3 1 • • • • • 2 4 • • • 4 2	1 2 3 • • 2 • 1 • • • 3 2 • • • • • 4 5 • • • 5 4	

Type $\begin{pmatrix} 2+2+2+2+3 \\ 2+2+2+5 \end{pmatrix}$ Possible non–isomorphic shapes
1 2 3 4 5 2 1 • • • 3 • 2 • • • • • 5 4

Type $\begin{pmatrix} 2+2+2+2+3 \\ 2+2+3+4 \end{pmatrix}$ Possible non–isomorphic shapes					
1 2 3 4 • 2 3 • • 1 • • 4 • 3 • • • 1 4	1 2 3 4 • 2 1 • • 3 3 • • • 1 • • 4 3 •	1 2 3 4 • 2 3 • • 1 4 • • 2 • • • 1 • 3	1 2 3 4 • 2 3 • • 1 • 4 • 3 • • • 1 • 3	1 2 3 4 • 2 1 • • • 3 • • 2 4 • • 4 • 3	1 2 3 4 • 2 1 • • • 3 • 4 • 2 • • • 2 4
1 2 3 4 • 2 3 1 • • 4 • • • 2 • • • 2 4	1 2 3 4 • 2 1 • • • • • 4 5 3 • • • 3 5	1 2 3 4 • 2 1 • • 5 5 • • • 1 • • 4 3 •	1 2 3 4 • 2 • • • 3 3 1 • • 2 • • 4 3 •	1 2 3 4 • 2 • 1 • 3 • 3 • • 2 • 4 • 3 •	1 2 3 4 • 2 1 • • • 4 • • • 2 • • 2 3 4
1 2 3 4 • 2 • • • 1 • 4 1 3 • • 1 • • 2					

Table 2: Latin interchanges of size 11

Type

$$\begin{pmatrix} 2+2+2+2+3 \\ 2+3+3+3 \end{pmatrix}$$

Possible non–isomorphic shapes

1 2 3 • • 2 1 • • • 3 • • 1 2 • • 2 3 1	1 2 3 • • 2 3 • 1 • • • 1 3 2 • • 2 • 3	1 2 3 • • 2 3 • 4 • 4 • 1 • 3 • • • 3 4	1 2 3 • • 2 3 • 1 • • • 1 3 4 • • 4 • 3	1 2 3 • • 2 3 • 4 • 4 • • • 1 • • 1 3 4	1 2 3 • • 2 1 • • • 3 • • 2 4 • • 2 4 3

Type

$$\begin{pmatrix} 2+2+2+2+3 \\ 2+4+5 \end{pmatrix}$$

Possible non–isomorphic shapes

	1 2 3 4 5 2 3 4 1 • 5 • • • 2	1 2 3 4 5 2 1 4 5 • • • 5 • 3	1 2 3 4 5 2 1 4 3 • 5 • • • 2	1 2 3 4 5 2 1 • • • 3 • 4 5 2	

Type

$$\begin{pmatrix} 2+2+2+2+3 \\ 3+3+5 \end{pmatrix}$$

Possible non–isomorphic shapes

1 2 3 4 5 2 3 1 • • 4 • • 5 2

Type

$$\begin{pmatrix} 2+2+2+2+3 \\ 3+4+4 \end{pmatrix}$$

Possible non–isomorphic shapes

	1 2 3 4 • 2 1 4 • 3 3 • • 2 4	1 2 3 4 • 2 3 4 • 1 • 1 • 3 4	1 2 3 4 • 2 3 4 • 5 5 • • 1 4	1 2 3 4 • 2 1 4 • 5 • • 5 3 4	

Type $\begin{pmatrix} 2+2+3+4 \\ 2+2+3+4 \end{pmatrix}$ Possible non–isomorphic shapes					
1 2 3 4 2 3 1 • 3 4 • • 4 • • 2	1 2 3 4 2 3 1 • 3 4 • • 4 • • 3	1 2 3 4 2 1 4 • 3 • 1 • 4 • • 2	1 2 3 4 3 1 2 • • • 1 2 • • 4 1	1 2 • 2 1 • • 3 2 1 4 4 3 • 1	1 2 3 4 2 1 5 • 5 • 1 • • • 4 3
1 2 3 4 2 5 • • 5 3 1 • • 4 • 3	1 2 3 4 2 1 • • • 5 1 3 • 4 • 5	1 2 3 4 2 3 1 • • • 5 1 • • 4 5			

Type $\begin{pmatrix} 2+2+3+4 \\ 2+3+3+3 \end{pmatrix}$ Possible non–isomorphic shapes					
	1 2 3 4 2 3 4 • 4 • • 1 • 1 • 3	1 2 3 4 2 3 • • 3 4 • 1 • • 1 3	1 2 3 4 2 1 4 • 3 • 2 • • 4 • 2	1 2 3 4 3 1 2 • • • 1 2 • 4 • 1	
	1 2 3 4 2 1 • • 3 • 1 2 • 4 • 1	1 2 3 4 2 5 • • • 4 • 5 5 • 1 3	1 2 3 4 2 5 • • 4 1 5 3 1 •	1 2 3 4 2 1 • • • 4 1 5 • • 5 3	

Type $\begin{pmatrix} 2+3+3+3 \\ 2+3+3+3 \end{pmatrix}$ Possible non–isomorphic shapes			
	1 2 3 • 2 3 1 • 3 1 • 2 • • 2 1	1 4 • 3 2 1 • • • 2 1 4 3 • 2 1	

Type $\begin{pmatrix} 2+3+3+3 \\ 3+4+4 \end{pmatrix}$ Possible non–isomorphic shapes				
	1 2 3 4 2 3 1 • 3 1 4 2	1 2 3 4 2 1 4 5 3 • 5 2	1 2 3 4 2 3 4 5 3 • 5 1	