2013 UQ/QAMT Problem Solving Competition - Year 9 & 10 Paper

All questions have equal value. Non-CAS calculators may be used.

Question 1

My internet connection is slow. The more files I download at once the slower it gets. At 1:00 pm I started downloading the three files simultaneously. At that point the time remaining to complete the downloads were given as 10, 8 and 4 minutes. At what time will the downloads all be finished?

Solution: $\boxed{22/3 = 7\frac{1}{3} \text{ minutes}}$ Let the total bandwidth be 1 unit (units of bytes/sec). Assume the bandwidth is divided equally among the 3 files. File 1 uses 1/3 of the bandwidth for 10 minutes, so its total size is 10/3. Similarly for files 2, 3 so the total amount to be downloaded is $10/3+8/3+4/3 = 22/3$. In effect, it takes just as long to completely download file 1, then 2, then 3, using all available bandwidth. The answer is just the mean of the 3 download times.

Question 2

If the number $x = 111\cdots 1$ consists of 2013 digits which are all equal to 1, what are the 3 middle digits of x^2?

Solution: $\boxed{565}$ Let $R_n = \overbrace{11\cdots 1}^{n}$ be the number with n decimal digits, all equal to 1. Thus $10^{n-1} < R_n < 2 \cdot 10^{n-1}$ so R_n^2 has $2n-1$ decimal digits.

We have $R_{n+1} = 10R_n + 1$, so $R_{n+1}^2 - 100R_n^2 = 20R_n + 1 = \overbrace{22\cdots 2}^{n}\,1$. Since R_{n+1}^2 has $2n+1$ decimal digits, the first n of them agree with the first n digits of R_n^2. The last $n+1$ digits of R_{n+1}^2 are shifted two places left and increased by 2.

If 2098 is incremented by 2222 it becomes 4320. Let β be a sequence of repeating blocks of the form 098765432 possibly with a partial block initially followed by a terminal 1; say $\beta = \cdots 09876543(2098)76543(2098)765432\ 1$. Then $100\beta + 222\cdots 221$ has digit sequence $ab\beta$ for some a and b.

This proves the following: Let $n \geqslant 10$. Write $n = 9m + r$ with $1 \leqslant r \leqslant 9$. Then R_n^2 has $2n-1$ decimal digits as follows:

$$R_n^2 = \overbrace{123456790}\ \overbrace{123456790}\ \cdots\ \overbrace{123456790}\ C_r\ \overbrace{098765432}\ \overbrace{098765432}\ \cdots\ \overbrace{098765432}\ 1.$$

Then there are m leading blocks 123456790 (note: no 8), a middle block C_r as in the table below, then m trailing blocks 098765432 (with no 1) and then a very last digit 1.

r	C_r
1	empty
2	1 2
3	12 32
4	123 432
5	1234 5432
6	12345 65432
7	123456 765432
8	1234567 8765432
9	12345678 98765432

(Total number of digits is $9m + 9m + 1 + |C_r| = 18m + 1 + 2(r-1) = 2n-1$.)

Hence for $n = 2013$, $m = 223$, $r = 6$, $C_r = 1234\overline{565}432$. The middle digit is the 2013th digit, which is the 6th digit of C_r, namely 6. So the middle 3 digits are 565.

There are other ways to see this. One can write $R_n = (10^n - 1)/9$. Then $R_n^2 = \frac{1}{81}(10^{2n} - 2 \cdot 10^n + 1)$. The decimal expansion of $\frac{1}{81}$ is $0.\overline{012345679}$. Now consider shifting this sequence by $2n$ and n places respectively.

Question 3

A cone has radius 1, height 6. A cube just fits inside the cone, with the bottom of the cube level with the bottom of the cone. What is the volume of the cube?

Solution: $\boxed{27/8}$ Draw vertical cross section through the cone, with the diameter along the x axis centered at the origin. The equation of the sloping side is $y = -6x + 6$ and one corner of the cube is at $(x/2, x)$ and also on this line. Solving, $x = 12/8 = 3/2$ so the volume is $\boxed{27/8}$.

Question 4

The factorial, $n!$, of a non-negative integer n is defined by

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 1,$$

with $0! = 1$. How many triples a, b, c of non-negative integers are there such that

$$(a!)(b!) = a! + b! + c!\ ?$$

Solution: $\boxed{a = b = 3,\ c = 4 \text{ is the only solution.}}$ Note that $m! \mid n!$ iff $m \leqslant n$. If $a = 0$ or 1 then $b! = 1 + b! + c! > 1 + b!$ contradiction. If $a = 2$ then $2 \cdot b! = 2 + b! + c!$ so $b! - c! = 2$, which is impossible, so $a \geqslant 3$. Wlog we may take $a \leqslant b$. Then $a!$ divides all terms except possibly $c!$ so $a! \mid c!$ and $a \leqslant c$ also. If $c \leqslant b$ then $(a!)(b!) \leqslant 3 \cdot b!$ so $a! \leqslant 3$, contradiction. Thus $3 \leqslant a \leqslant b < c$. Hence $b!$ divides all terms except possibly $a!$, so $b! \mid a!$ so $a = b$.

Thus $c! = a!(a! - 2)$. Since $3 \mid a!$, the exact power of 3 dividing the LHS and RHS is the same, so there is no multiple of 3 in the range $a+1, \ldots, c$, so $c = a+1$ or $a+2$ only.

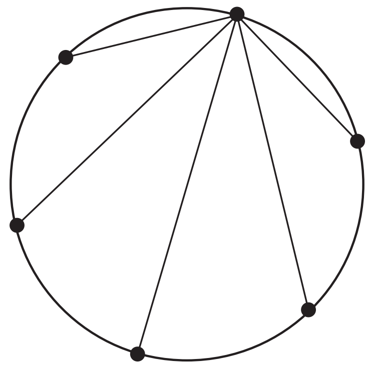
If $c = a+1$, $a + 1 = a! - 2$ so $a! - a = 3$ so $a \mid 3$ so $a = b = 3$, $c = 4$ and this works.

If $c = a+2$, $a^2 + 3a + 2 = (a+2)(a+1) = a! - 2$, so $a! - a^2 - 3a = 4$ so $a \mid 4$ with $a \geqslant 3$, so $a = b = 4$, $c = 5$, but this does not work.

So the only solution is $a = b = 3$, $c = 4$.

Question 5

Suppose we have six equally spaced points on a circle of radius 1 unit. We connect one point to each of the other five with straight lines, as in the diagram. What is the product of the lengths of all these lines?



Solution: $\boxed{6}$ There's a very easy solution using complex numbers. Let $\zeta = e^{2\pi i/n}$ for n points. We want $|1-\zeta| \cdot |1-\zeta^2| \cdots |1-\zeta^{n-1}|$. This is just the absolute value of the cyclotomic polynomial $(z^n - 1)/(z-1) = 1 + z + \cdots + z^{n-1}$ evaluated at 1, which is obviously n.

So for 6 points, the product of lengths is 6. This could be found be fairly easy geometry. Changing n changes the geometrical difficulty.