THE UNIVERSITY OF QUEENSLAND
AUSTRALIA

2012 UQ/QAMT Problem Solving Competition - Year 9 & 10 Paper

All questions have equal value.

Question 1

If a is a real number, $\lfloor a \rfloor$ denotes the largest integer n with $n \leqslant a$, and $\{a\}$ denotes the number r with $0 \leqslant r < 1$ for which $a - r$ is an integer. For example $\lfloor \frac{9}{4} \rfloor = 2$ and $\{\frac{9}{4}\} = \frac{1}{4}$. Which values of x (if any) satisfy

$$2\lfloor x \rfloor = x + 2\{x\}\ ?$$

Solution: $\boxed{x=0, 4/3, 8/3}$ We always have $x = \lfloor x \rfloor + \{x\}$, so the required equation can be written $2\lfloor x \rfloor = \lfloor x \rfloor + 3\{x\}$, so $\lfloor x \rfloor = 3\{x\}$. The LHS is an integer, and $0 \leqslant 3\{x\} < 3$, so $\lfloor x \rfloor$ and $3\{x\}$ are each 0, 1 or 2. That is, $x = 0, 1$, or $2 + 0, 1/3$ or $2/3$. So x can only be $0/3, 1/3, \ldots, 7/3$ or $8/3$. Checking the 9 possibilities leads to the given solution.

Question 2

Starting with a positive integer n, form the sum of decimal digits of n, then form the sum of digits of this new number and so on, until the process stabilizes. The result is called the *ultimate digital sum* of n. What is the ultimate digital sum of 2^{2012}?

Solution: $\boxed{4}$ The ultimate digital sum of n is just the smallest non-zero remainder after n is divided by 9. See Year 8 for more details.

If n has remainder r after division by 9 then $n = 9q + r$, so $2n = 9(2q) + (2r)$ has remainder $2r$, where remainders greater than 9 are reduced by 1 more multiple of 9 so that they lie in the range $1, 2, \ldots 9$.

For powers of 2, each remainder is thus twice the previous one (taken modulo 9), starting with $\boxed{2, 4, 8, 7, 5, 1.}$ The next remainder will then be twice the previous one, namely 2. Thus these 6 remainders are repeated in a cycle.

$2012 = 6(335) + 2$ so calculating 2^{2012} we run through 335 complete cycles, and then 2 entries into the next cycle, ending at 4.

Question 3

A car, van, truck and bike are all travelling in the same direction on the road. Each travels at a constant speed, but the speeds of the 4 vehicles may be different. At 10 am the car overtakes the van. At noon it overtakes the truck. At 2 pm it overtakes the bike. At 4 pm the truck overtakes the bike. At 6 pm the van overtakes the truck. When does the van overtake the bike?

Solution: $\boxed{\text{4:24 pm}}$ At 10 am the car and van are at the same position, with the truck some distance t ahead, and the bike distance b ahead. We only need to consider the relative speeds of the vehicles, so we can assume the bike is stationary (we can imagine we are the bike rider, watching the event and noting the time vehicles cross). Choose units such the speed of the truck is 1. Let the speeds of the car and van be c and v. We need to find time T with $Tv = b$, so we need to find $T = b/v$. The given information is:

$$\begin{aligned} 2c &= t + 2 \\ 4c &= b \\ 6 + t &= b \\ 8v &= t + 8 \end{aligned}$$

Solving this system $(b, c, t, v) = (8, 2, 2, 5/4)$ so $T = b/v = 32/5$ hours after 10 am, or 4:24 pm.

Question 4

Two congruent non-overlapping equilateral triangles are placed wholly within a square of side length 1. What is the maximum area they could cover?

Solution: $\boxed{1/\sqrt{3}}$ Let the side length of the equilateral triangles be s. If the triangles do not touch or do not touch the square it would be possible to increase their size and cover more area.

One finds that the optimal configuration is that the two triangles each have one vertex on diagonally opposite corners of the square, with shared base along a diagonal of the square.

The diagonals have length $\sqrt{2}$ by Pythagoras, and divide the area covered into 4 smaller right angled triangles with sides $s/2$, $\sqrt{2}/2$ and s. Applying Pythagoras, $s = \sqrt{2/3}$, while the total area covered is $4 \cdot \frac{1}{2}\frac{s}{2}\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{3}}$ (about 58% of the square).

Question 5

Suppose x, y and z are numbers satisfying

$$\begin{aligned} x + y + z &= -2 \\ x^2 + y^2 + z^2 &= 122 \\ x^3 + y^3 + z^3 &= 142 \end{aligned}$$

What is the value of xyz?

Solution: $\boxed{168}$ Rather than trying to solve for x, y and z [In fact, $x = -3$, $y = -7$, $z = 8$ works] we instead aim to write xyz in terms of $x + y + z$, $x^2 + y^2 + z^2$ and $x^3 + y^3 + z^3$.

Consider $(x+y+z)^3 - (x^3+y^3+z^3) = 3(x^2y+x^2z+xy^2+xz^2+y^2z+yz^2)+6xyz = 3A+6xy$, say. The term A looks like the cross terms obtained from $(x+y+z)(x^2+y^2+z^2)$, and in fact $(x+y+z)(x^2+y^2+z^2) = x^3+y^3+z^3+3A$.

In summary, if $x+y+z = p_1 = -2$, $x^2+y^2+z^2 = p_2 = 122$ and $x^3+y^3+z^3 = p_3 = 142$, then

$$xyz = \frac{1}{6}\left(p_1^3 - 3p_1p_2 + 2p_3\right) = 168.$$

This is an example of *Newton's identities.*