THE UNIVERSITY OF QUEENSLAND
AUSTRALIA

2006 QAMT Problem-Solving Competition - Year 9 & 10 Paper

All questions have equal value.

Question 1
You lay out a 100 m rope flat on the ground in a straight line, anchored at both ends. Now suppose you play out 1 m of slack, so the 101m rope is anchored at both ends, but can be lifted slightly off the ground at the 50 m mark. How high can it be lifted?

Solution $\boxed{7.1\text{m}}$ We use Pythagoras: in a right angle triangle with hypotenuse 50.5 and side length 50, what is the length of the third side? It is $\sqrt{50.5^2 - 50^2} = \sqrt{201}/2 \simeq 7.1\text{m}$. *In general if* ϵ *rope is played out the height will be* $\sqrt{(x/2+\epsilon/2)^2 - (x/2)^2} \simeq \sqrt{x\epsilon/2} \simeq 7.1\sqrt{\epsilon}$ *for 100 m of rope. So even playing out 1cm extra of slack gives height about 71 cm! Or: if you give as much slack as the length of your middle finger, you would be able to walk under the rope at the centre.*

Question 2
You have a 20 cm strip of paper that is 1 cm wide, and lots of 1 cm and 2 cm pieces of tape, also 1 cm wide. How many different ways can you cover the paper with the tape (without overlap)?

Solution $\boxed{6765.}$ Let F_n be the number of ways to cover a strip of length n. Consider F_{n+1}. Beginning at the left corner there must be a strip of length 1 or 2, leaving n or $n-1$ cm still to cover, and this can be done in F_n or F_{n-1} ways respectively. So $F_{n+1} = F_n + F_{n-1}$ with $F_1 = 1$ and $F_2 = 1$, so the answer is the Fibonacci numbers. By repeated addition we find the Fibonacci numbers are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765 so $F_{20} = 6765$. Note that $F_{20}/F_{19} = 1.61803396\ldots \simeq 1.6180339887\ldots = (\sqrt{5}+1)/2$, the *golden ratio.*

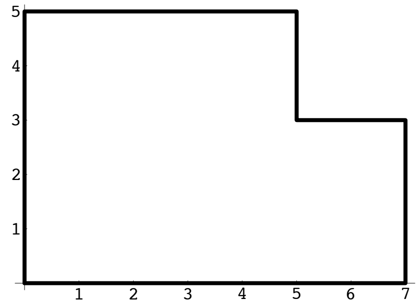
Question 3
In the rectangle ABCD a triangle AEF is drawn, where E lies on BC and F lies on CD, and the triangles ABE, ECF and ADF all have the same area. What is the ratio CF:FD?

Solution $\boxed{(\sqrt{5}+1)/2}$ Without loss of generality let AB=1 and AD=r (fixed). Let FD= y and EC= z. Then we need $ry = (1-y)z = r - z$. So $z = r(1-y)$ so $ry = (1-y)r(1-y)$ ie $(1-y)^2 - y = 0$ so $y^2 - 3y + 1 = 0$, so $y = (3 \pm \sqrt{5})/2$ with $0 < y < 1$ and so the ratio is $(1-y)/y = 1/y - 1 = 2/(3-\sqrt{5}) - 1 = (3+\sqrt{5})/2 - 1 = (\sqrt{5}+1)/2$. So the ratio is $\boxed{\frac{\sqrt{5}+1}{2}}$.
(This is the *golden ratio.*)

Question 4
In front of you is a pile of coins, containing between 1500 and 3000 coins. If you divide it into groups of 7, 11 or 13 each time you have 4 coins left over. How many coins are in the pile?

Solution $\boxed{2006}$ The obvious solution is 4 coin, but this impossible since we have at least 100 coins. If we add any multiple of $7 \cdot 11 \cdot 13 = 1001$, this does not change the remainder upon dividing by 7, 11 or 13, so $2002 + 4$ is a solution. Alternative solution: let n be the number of coins. Then $n = 7x + 4 = 11y + 4$ for some integers x and y so $7x = 11y$ so x must be divisible by 11. Thus $n = 77z + 4$. But also $n = 13w + 4$ for some w, so now z is divisible by 13, and we get $x = 1001t + 4$ for some t. To finish with between 1500 and 3000 coins the only possible t we can take is 2, leading to 2006 coins. It is easy to check that this does indeed give a solution.

Question 5
Consider the region in the plane shown in the following diagram. A line $y = mx$ splits the region into two subregions of equal area. What is the value of m?



Solution The total area is 31. The height of the line when $x = 5$ is $5m$ so the area of the triangle is $\frac{1}{2} \times 5 \times 5m = \frac{25}{2}m$, giving a total area in the lower part of $6 + \frac{25}{2}m$. Solving this equal to $\frac{31}{2}$ gives $m = \frac{19}{25} = 0.76$.

Question 6
Tamref's First Theorem concerns the equation $n^x + n^y = n^z$ where n, x, y, z are all positive integers. Describe all solutions of this equation.

Solution Write down the equation in base n. Then $1\overbrace{0\cdots0}^{x}+1\overbrace{0\cdots0}^{y} = 1\overbrace{0\cdots0}^{z}$ and the only way this can happen is if $x = y$ and $n = 2$. So $2^x + 2^x = 2^{x+1}$ (for any x) is the only solution.