THE UNIVERSITY OF QUEENSLAND
AUSTRALIA

2005 QAMT Problem-Solving Competition - Year 9 & 10 Paper

Question 1 Let a and b be integers, with $0 \leqslant a, b \leqslant 9$ and $a \neq 0$. Find all 6 digit numbers whose decimal representation is $a2005b$ that are exactly divisible by 13.

2 marks

Solution
Let $N = a2005b = a \cdot 10^5 + 20050 + b = a(7692 \cdot 13 + 4) + (1542 \cdot 13 + 4) + b$. Then N is divisible by 13 exactly when the number $r = 4a + 4 + b = 4(a + 1) + b$ is. Now $4 \leqslant r \leqslant 49$ and so the possible multiples of 13 r could be are 13, 26, 39. If $r = 13$ then $a = 1$ and $b = 5$ or $a = 2$ and $b = 1$. Etc.
The numbers are: 120055, 220051, 420056, 520052, 720057, 820053.

Question 2 According to a survey, at least 70% of people like apples, at least 75% like bananas and at least 80% like cherries. What can you say about the percentage of people who like all three?

4 marks

Solution
Let a be the percentage who like only apples, b the percentage who like only bananas, ab the percentage who like apples and bananas but not cherries and so on. (ab is not the product $a \cdot b$.) We want to maximize/minimize abc given constraints:

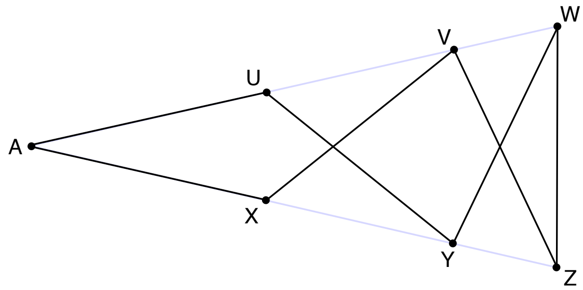
$$\begin{aligned}
a, b, c, ab, ac, bc, abc &\geqslant 0 \\
a + b + c + ab + ac + bc + abc &\leqslant 100 \\
a + ab + ac + abc &\geqslant 70 \\
b + ab + bc + abc &\geqslant 75 \\
c + ac + bc + abc &\geqslant 80
\end{aligned}$$

(See Venn diagram). The first and second inequalities imply $abc \leqslant 100$ and clearly $abc = 100$ and all other variables $= 0$ is a valid solution.
Adding the last 3 inequalities and subtracting twice the second line gives $abc \geqslant 25 + (a+b+c) \geqslant 25$. Taking $ab = 20$, $ac = 25$, $bc = 30$, $abc = 25$ shows $abc = 25$ is possible. Thus

$$25 \leqslant abc \leqslant 100.$$

Problems of this type are called *linear programming* problems.

Question 3 You have 7 matches of equal length, AX, XV, VZ, ZW, WY, YU and UA laid out as in the diagram. Find the angle at A.



3 marks

Solution
Let the angle at A be θ. Triangles AXV is isosceles, so angle AVX is θ, so angle VXY is 2θ. Triangle XVZ is isosceles, so angle YZV is 2θ and so angle ZVA is $\pi - 3\theta$, so ZVW is 3θ. Since triangle ZVW is isosceles, angle VWZ is also 3θ. By symmetry, angle YZW is 3θ, so the angles in triangle AZW sum to 7θ. Hence theta is $\pi/7$ radians, or about $26°$. This gives a matchstick "construction" of a regular heptagon.

Question 4 Find all solutions of the equation

$$y^2 = x^6 + 17$$

where x and y are integers.

3 marks

Solution
If (x, y) is a solution so are $(\pm x, \pm y)$ so we may consider only $x, y > 0$. Now $y^2 - x^6 = 17$ so $(y - x^3)(y + x^3) = 17$. There are four possibilities: $y - x^3 = 1$, $y + x^3 = 17$ or $y - x^3 = -1$, $y + x^3 = -17$, or $y - x^3 = 17$, $y + x^3 = 1$ or $y - x^3 = -17$, $y + x^3 = -1$. Since $x, y > 0$, the second and fourth cases are eliminated.
In the first case, adding the two equations gives $2y = 18$ so $y = 9$ and $x = 2$. In the third, $y = 9$ again but this leads to negative x so we can ignore it. Thus $(x, y) = (\pm 2, \pm 9)$ are the only solutions.

Question 5 You have one rectangle of each of the following dimensions: 1×18, 2×16, 3×13, 4×11, 5×10, 6×9 and 7×7. Is it possible to arrange them into a single large rectangle? Explain.

4 marks

Solution
The total area of the small rectangles is $2 \cdot 11 \cdot 13$. The only possible dimensions for the large rectangle that will fit the 7×7 square inside are 22×13 or 26×11. In the first case, the 1×18 and 2×16 rectangle must be parallel to the longer side. This suggests placing the 1×18 and 4×11 pieces next to each other with the 2×16 piece above. The rest then fills in easily. See diagram.

Question 6 Alice, Bob and Cathy take turns (in that order) in rolling a six sided die. If Alice ever rolls a 1, 2 or 3 she wins. If Bob rolls a 4 or a 5 he wins, and Cathy wins if she rolls a 6. What is the probability that Cathy wins?

4 marks

Solution
Let the probability that Cathy wins be p. The probability Alice wins on her first roll is $1/2$. The probability Bob wins on his first roll is $1/2 \cdot 1/3 = 1/6$ (there is only a $1/2$ chance Alice has not won, so that he gets to roll at all). The probability Cathy wins on her first roll is $(1 - 1/2 - 1/6) \cdot 1/6 = 1/18$. The probability that no one wins in the first three rolls is $\frac{3}{6} \cdot \frac{4}{6} \cdot \frac{5}{6}$. After that the

game effectively starts again, so

$$p = \frac{1}{18} + \frac{5}{18}p$$

which gives $p = 1/13$.
Alternatively, p is given by

$$p = \frac{1}{18}\left(1 + \frac{5}{18} + \left(\frac{5}{18}\right)^2 + \cdots\right) = \frac{1}{18} \cdot \frac{1}{(1 - \frac{5}{18})} = \frac{1}{13}.$$