

2004 QAMT Problem-Solving Competition - Year 9 & 10 Solutions

Question 1 I have 189 dollar coins and 20 jars. I want to distribute the coins amongst the jars so that each jar has a different number of dollars in it (note that I allow \$0 in a jar). Can I do it? If so, tell me how. If not, explain why not? 2 marks

Solution It cannot be done. The smallest amount of money I can distribute in this way is $0 + 1 + \dots + 19 = 190$.

Question 2 The values of a , b , and c are such that $a - b = b - c = 8$. Determine the numerical value of $a^2 - 2b^2 + c^2$. 2 marks

Solution 128.

Various algebraic ways to do this. Eg $a = b + 8$, $c = b - 8$ so $a^2 - 2b^2 + c^2 = 128$.

Question 3 Find a four digit even number $abcd$ (where a, b, c, d are the four digits) such that 1.5 times $abcd$ equals the four digit number $dcba$. ($a = 0$ is not allowed.) 4 marks

Solution 6534

Let n be the number. d is even and not 0, so $d \in \{2, 4, 6, 8\}$ while $a < 7$ (else $1.5n$ has five digits). The condition is $abcd + abcd + abcd = dcba + dcba$. The last digit of $3d$ is the same as the last digit of $2a$. Let $3d - 2a = 10z$. Then $10z < 3d < 30$ and $10z > -2a > -14$, so $z \in \{-1, 0, 1, 2\}$. Next write $3a + x = 2d + y$ where x and y represent possible carries: $x \in \{0, 1, 2\}$, $y \in \{0, 1\}$. Multiplying by 2, $4d + 2y = 6a + x = 9d - 30z$ so $5(d - 6z) = 2(y - x)$. Hence 5 divides $y - x$, so $y = x$ and $d = 6z$. Since d is a non-zero digit we have $d = 6$. Now $2dcba$ is between 12000 and 14000, so $a = 4$. Looking at the middle digits we need $2cb = 3bc + 1$, ie $20c + 2b = 30b + 3c + 1$, so $17c = 28b + 1$, so c is odd. Testing $c = 1, 3, 5, 7, 9$, the only possibility is $b = 4$, $c = 5$. So $n = 4356$. Check: $1.5 \cdot 4356 = 6534$.

Question 4 Two planets are orbiting a star in circular orbits. The first takes 12 years to make an entire orbit, and the second takes 32 years. Currently both planets and the star are aligned along a straight line. How many years pass until this next happens? (The planets do not have to return to their starting positions, they just need to line up.) 4 marks

Solution $96/5 = 19.2$ years.

The first planet travels 2π radians in 12 years, at an angular velocity of $2\pi/12$ radians per year. After t years it is at angle $t \cdot 2\pi/12$. So the condition for the two planets to line up is $t \cdot 2\pi/12 = 2k\pi + t \cdot 2\pi/32$, where k is an integer indicating the number of "extra laps" the faster planet has done. Solving, $t = 96k/5$, so every $96/5 = 19.2$ years.

Slick proof: the lcm of 12 and 32 is 96, corresponding to 8 orbits of the faster planet and 3 of the slower, so the faster must overtake the slower 5 times during these 96 years. Such occurrences are regularly spaced, so the time between them is $96/5$ years. This is every 216° .

Question 5 A belt is drawn tight around 3 circles of radius 10 cm, as shown. How long is the belt? 4 marks

Solution $20\pi + 60$ cm.

The belt consists of 3 arcs and 3 straight lines (otherwise it could be drawn tighter). Connect the centres of the triangles. Draw radii to the point at which the belt leaves the circles. The belt here is tangential, so the angle between the radii and the belt is 90° . It follows that the radii, belt and triangle sides form rectangles. Also the angle from one radii to the next is 120° , so the belt wraps around one third of each circle.

Thus the total length is the perimeter of one circle, plus three times the side length of the triangle: $2\pi r + 6r$ or $20\pi + 60$ cm.

Question 6 Fill in the 3 by 3 square below with positive integers in such a way that the *product* of entries in each row, column and the two diagonals is 1000. No integer appearing in the square may occur more than once. 4 marks

Solution $1000 = 2^3 \cdot 5^3$ so the factors of 1000 are all of the form $2^a \cdot 5^b$, with $a, b = 0, 1, 2, 3$, so we may view the problem as one of finding a magic square of 9 distinct pairs (a, b) of this form. The multiplicative condition is that the sum of a 's from in any row, column or diagonal is 3, and similarly for the b 's. Thus the problem is solved if we can find two 3 by 3 (additive) magic squares, all of whose entries are 0, 1, 2 or 3, such that if corresponding entries in the two squares are paired, 9 distinct pairs are obtained. It's easy to find such a square. For example

$$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}.$$

The mirror image is another example, and these two squares give rise to 9 distinct pairs:

$$\begin{pmatrix} (2,1) & (0,0) & (1,2) \\ (0,2) & (1,1) & (2,0) \\ (1,0) & (2,2) & (0,1) \end{pmatrix} \rightarrow \begin{pmatrix} 20 & 1 & 50 \\ 25 & 10 & 4 \\ 2 & 100 & 5 \end{pmatrix}.$$