2003 QAMT Competition
Year 9&10 SOLUTIONS

Question 1. How many times in one day do the hands of a clock form a right angle? 1 mark

A 22 **B** 24 **C** 44 **D** 46 **E** 48

Answer: C, 44

Each time the hands cross they make two right angles. In one day, they cross 22 times (two twelve hour periods). This makes 44 right angles in one day.

Question 2. The digits 1234 are written in increasing order. You may insert any number of plus or minus signs between any of the digits to produce an answer. For example 1 mark

1 – 2 + 34 gives the answer 33.

How many distinct positive answers can you attain?

A 4 **B** 12 **C** 17 **D** 26 **E** 27

Answer: (C) 17

Solution: With a system, you can enumerate all of the cases. These are the ones picked out with positive results.

1-2+3+4=6

1+2-3+4=4

1+2+3-4=2

1+2+3+4=10

12-3-4=5

12-3+4=13

12+3-4=11

12+3+4=19

12+34=46

1+23-4=20

1+23+4=28

1-2+34=33

1+2+34=37

123-4=119

123+4=127

1+234=235

1234

Note all of the right hand sides are different. So there are 17 altogether (C)

Question 3. If numbers a,b,c satisfy $a+b+c = 0$ and $a^2+b^2+c^2 = 1$ then what is $a^4+b^4+c^4$?

1 mark

A 1/4 **B** 1/2 **C** 1 **D** 4 **E** None of these

Answer: B, ½

Solution: Can check with some sample values to confirm the answer is B. Note that $a^2+b^2+c^2=1$ means a,b,c must lie between –1 and +1.

Given, $a+b+c=0, \quad a^2+b^2+c^2=1$ so

$$\left(a^2+b^2+c^2\right)^2=1$$
$$\Rightarrow a^4+b^4+c^4+2a^2b^2+2b^2c^2+2a^2c^2=1$$
$$\Rightarrow a^4+b^4+c^4=1-2\left(a^2b^2+b^2c^2+a^2c^2\right)$$

so the result (B) will hold if we can show that $\left(a^2b^2+b^2c^2+a^2c^2\right)=\frac{1}{4}$ (*).

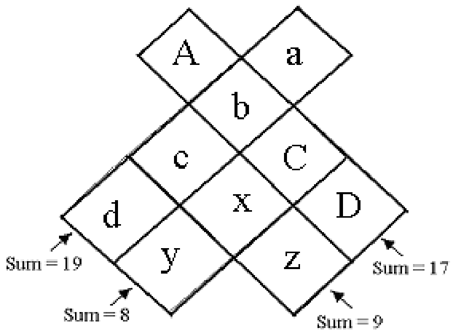
Now, $a+b+c=0$, so $(a+b+c)^2=0$. Hence

$$a^2+b^2+c^2+2ab+2bc+2ac=0$$
$$\Rightarrow 1+2(ab+bc+ac)=0$$
$$\Rightarrow ab+bc+ac=-\frac{1}{2}$$
$$\Rightarrow (ab+bc+ac)^2=\frac{1}{4}$$
$$\Rightarrow (ab)^2+(bc)^2+(ac)^2+2ab^2c+2a^2bc+2bac^2=\frac{1}{4}$$
$$\Rightarrow a^2b^2+b^2c^2+a^2c^2+2abc(a+b+c)=\frac{1}{4}$$
$$\Rightarrow a^2b^2+b^2c^2+a^2c^2=\frac{1}{4}$$

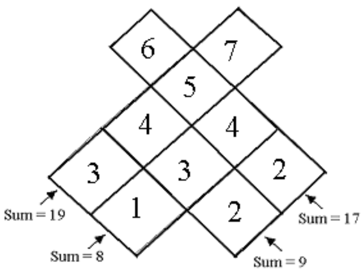
confirming (*) and so the result (B) holds.

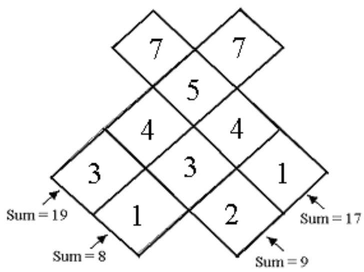
2 marks

Question 4. Whole numbers from 1 to 7 are to be placed in the cells of this figure. Each number may be used once, twice or not at all. The numbers in the four sloping rows shown must have the sum given and the numbers must decrease in size going down each of these four rows. Find all ways in which this can be done.



Solution: 2 ways.





Let the numbers in the cells be as shown, so the numbers must decrease with increasing alphabetical order, except for d and y or D and z. Consier $a+b+c+d=19$. Since $6+5+4+3=18<19$, we must have $a=7$. Since $7+4+3+2=16<19$, there are only two possible choices for b: $b=6$ or $b=5$.

Case 1 ($b=6$). Then $A=7$. From $A+b+C+D=17$, we have $C+D=4$, so $C\leq 3$. But then $C+x+y\leq 3+2+1=6$, whereas we must have $C+x+y=8$. So this case is impossible.

Case 2 (b=5). Then from a+b+c+d=19, we must have $c=4$ and $d=3$. Since $C<b$, we know $C\leq 4$. Because $C+x+y=8$ and $3+2+1=6$, we must have C=4. And since $4+2+1=7<8$, we must also have $x=3$ and $y=1$. From $c+x+z=9$, we get $z=2$. Only A and D remain, with $A+b+C+D=17$ their only requirement. So we can have $A=7$ and $D=1$ or $A=6$ and $D=2$. These are the two solutions shown.

Question 5. Start with any three digit number in which the hundreds digit is at least two more than the ones digit. Reverse the digits. Subtract the smaller number from the larger to obtain their difference. Now reverse the digits of the difference. Add this number to the difference. Prove that, no matter what your starting number was, the result is always 1089. 2 marks

Solution:

For example, if you start with 521, you find:

521 – 125 = 396 and 396 + 693 = 1089.

Let the number have digits a,b,c, so the number is 100a + 10b + c and a ≥ c + 2. Its reverse is 100c + 10b + a, which is smaller than the original number, so subtracting gives the difference as 100(a-c-1) + 90 + (10+c-a). Note c<a, so you must borrow in the units column, and then you must borrow to subtract in the tens column. Reversing this gives 100(10+c-a) + 90 + (a-c-1). So adding the difference and its reverse gives 100(10-1) + 180 + (10-1), ie, 1089 as required.

Question 6: A vault is set to open in some number of years into the future. The control panel on the vault is a series of switches, numbered from 1 to 10 (from left to right). Each switch is set either to 1 or 0. The year that the vault will open is coded into the switches. 3 marks

If the *n*th switch is "on", it means that the doors will stay shut for a further 2^{n-1} years if n is even, or 2^{n-1} fewer years if n is odd. If a switch is set to zero there is no effect. How should the switches be set to make the vault open in 200 years time?

Answer: 0001001011

Solution:

Note that $200 = 128 + 64 + 8 = 2^7 + 2^6 + 2^3$.

However, even powers must have coefficients of +1, not –1. Using the fact that $2^6 = 2^7 - 2^6$, have that $200 = 2^7 + (2^7 - 2^6) + 2^3 = 2.2^7 - 2^6 + 2^3 = 2^8 - 2^6 + 2^3$.

Similarly, use the fact that $2^8 = 2^9 - 2^8$ to get

$200 = 2^{10-1} - 2^{9-1} - 2^{7-1} + 2^{4-1}$

and so switches 4, 7, 9 and 10 must be set to "on".