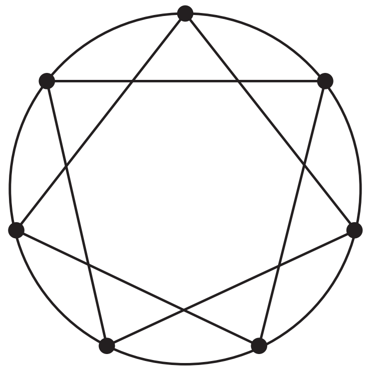
2013 UQ/QAMT Problem Solving Competition - Year 8 Paper

All questions have equal value. Non-CAS calculators may be used.

Question 1
Consider the star made by connecting 7 points equally spaced around the circumference of a circle, as shown in the diagram. What is the interior angle of the heptagon?



Solution: $\boxed{77\frac{1}{7}^\circ}$ If the interior angle is θ then the exterior angle is $180-\theta$. 7 copies of this give two full rotations, so $7(180-\theta) = 720$, so $\theta = 540/7^\circ$.

Question 2
At some point in 2011 or 2012 Michael suddenly observed "at the next exact hour we'll have three times as many minutes remaining this month as there will have been hours in the past part of the month." What was the exact date (to the nearest hour)?

Solution: $\boxed{\text{4 pm on 27th Feb, 2011}}$ Any non-leap year works.
Let $m \in \{28, 29, 30, 31\}$ be the month length in days, let d be the number of complete days in the month elapsed so far, $0 \leqslant d \leqslant m-1$, let h be the hour number of Michael's observation, $0 \leqslant h \leqslant 23$. The observation is

$$3(24d + h) = (24 - h)60 + (m - d - 1)(24)(60).$$

That is,

$$160m = 7(24d + h).$$

It follows that 7 is a factor of m, so $m = 28$ and the month is February, and the year is not a leap year, so not 2012. Thus $0 \leqslant h = 8(80 - 3d) < 24$ and so $80 - 3d \in \{0, 1, 2\}$. It follows that $d = 26$ and $h = 16$.

Question 3
My internet connection is slow. The more files I download at once the slower it gets. At 1:00 pm I started downloading the three files simultaneously. At that point the time remaining to complete the downloads were given as 10, 8 and 4 minutes. At what time will the downloads all be finished?

Solution: $\boxed{22/3 = 7\frac{1}{3} \text{ minutes}}$ Let the total bandwidth be 1 unit (units of bytes/sec). Assume the bandwidth is divided equally among the 3 files. File 1 uses 1/3 of the bandwidth for 10 minutes, so its total size is 10/3. Similarly for files 2, 3 so the total amount to be downloaded is $10/3+8/3+4/3 = 22/3$. In effect, it takes just as long to completely download file 1, then 2, then 3, using all available bandwidth. The answer is just the mean of the 3 download times.

Question 4
A chessboard has squares of side length 1 unit. A circle is drawn through the four corners of each black square and its interior coloured black (so each black square is covered up). What is the area of the remaining white space?

Solution: $\boxed{60 - 14\pi \simeq 16.02 \text{ units}^2 \text{ or almost exactly 1/4 of the board's area.}}$ Let r be the radius of the circles. By Pythagoras, $r^2 + r^2 = 1$, so $r = 1/\sqrt{2}$, and each has area $\pi/2$. There are 32 white squares, consisting of 2 corners, 12 edges and 18 central squares, so $2 \cdot 2 + 12 \cdot 3 + 18 \cdot 4 = 112$ curved sections are removed (note that they do not overlap). The area of each is $\frac{1}{4}\frac{\pi}{2} - \frac{1}{2}\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} = \frac{\pi-2}{8}$, so the remaining white area is $32 - 14(\pi - 2) = 60 - 14\pi \simeq 16.02$ units^2. Almost exactly 3/4 of the board (in fact 74.97%) is now black.

Question 5

An eggsellent merchant sells eggs. One day a customer walked up and said, "I'll buy half of all your eggs, plus half an egg." The merchant sold the appropriate number of eggs. Soon after that another customer came along and said exactly the same thing, and again bought the appropriate number of eggs. A third customer and fourth customer in turn did the same. At this point the eggsellent merchant had sold all his eggs. How many did he start with?

Solution: $\boxed{15}$ Let x_i ("eggxs") be the ("eggxact") number of eggs left after serving i customers, $i = 0, 1, \ldots$. Thus

$$x_{i+1} = x_i - \left(\frac{1}{2}x_i + \frac{1}{2}\right), \qquad x_4 = 0$$

and we need to find x_0. Thus $x_i = 2x_{i+1} + 1$. So $x_3 = 1$, $x_2 = 3$, $x_1 = 7$, $x_0 = 15$.

Note that these are of the form $2^n - 1$. With 4 customers the initial number of eggs would be 15, which is not prime. With 5 customers it would be 31, which is, so 5 customers is the minimum.