2012 UQ/QAMT Problem Solving Competition - Year 8 Paper

All questions have equal value.

Question 1

Starting with a positive integer n, form the sum of decimal digits of n, then form the sum of digits of this new number and so on, until the process stabilizes. The result is called the *ultimate digital sum* of n. How many integers in the range 1, 2, 3, ..., 2012 have ultimate digital sum equal to 7?

Solution: $\boxed{223}$ Let $s(n)$ be the sum of decimal digits of n. That is, if $n = a_0 + a_1 \cdot 10 + a_2 \cdot 10^2 + \cdots$ then $s(n) = 1_0 + a_1 + a_2 + \cdots$. So $n - s(n) = a_1(10-1) + a_2(10^2-1) + \cdots$, which is divisible by 9. So n, $s(n)$, $s(s(n))$, ... all leave the same remainder after division by 9. It follows that for any positive integer n, *the ultimate digital sum of n is just the smallest non-zero remainder after n is divided by* 9. Thus, integers 1 to 9 have ultimate digital sum 1 to 9 respectively, after which the cycle repeats.

$2012 = 9(223) + 5$ which means there are 224 numbers with ultimate digital sum 1, 2, 3, 4, 5 and 223 with 6, 7, 8, 9.

Question 2

George Orwell's novel "1984" begins

It was a bright cold day in April and the clocks were striking thirteen.

The year 1984 is unusual in that there were 5 Wednesdays in February, and exactly three Friday the 13ths in the year, the maximum possible. What is the next such year?

Solution: $\boxed{2012}$ Of the two conditions, the February condition is far more restrictive. In non-leap years February has 28 days, which is exactly 4 weeks, so each day of the week occurs four times. Five Wednesdays occur in February if (and only if) February 1st is a Wednesday, and the year is a leap year.

A non-leap year has $365 = 7(52) + 1$ days, so February 1st in the year $y + 1$ will be 1 week day later than February 1st in the year y, or 2 week days later if y is a leap year.

We know February 1st 1984 was a Wednesday, so it was a Friday in 1985 and so on. The years in which February 1st was a Wednesday are: 1984, 1989, 1995, 2006, 2012, ... (5 years later if the interval contains 2 leap years; 6 years later otherwise; except in a leap year the Wednesday can be missed altogether as happened in 2000). So 2012 is the first year after 1984 in which there were 5 Wednesdays in February.

We must check if 2012 has 3 Friday 13ths. And indeed it does: in January, April and July. There are various ways to see this. A month having a Friday 13th is equivalent to the 1st of the month being Sunday. If the 1st day of a month of length ℓ is the dth day of the week, the 1st of the next month will be the $d+1$st day if $\ell = 29$, $d+2$nd day if $\ell = 30$ and $(d+3)$rd day if $\ell = 31$. This allows us to check all 12 months in 2012 easily.

More calculation shows that the years since 1800 in which there were five Wednesdays in February were: 1832, 1860, 1888, 1928, 1956, 1984, 2012, 2040, ... The 13-full years starting from 1981 are: 1981, 1984, 1987, 1998, 2009, 2012, 2015, 2026, 2037, 2040, 2043, ...

Question 3

How many positive integer solutions x and y are there of the equation $x^2 - y^2 = 75$?

Solution: $\boxed{3 \text{ solutions } (x, y)}$ Since $x^2 - y^2 > 0$ with x and y positive we have $x > y$. Factorizing, $(x - y)(x + y) = 75$ where the two factors on the left are positive integer factors of 75. Given $x - y$ and $x + y$, we can find x by adding and dividing by 2: $x = \frac{1}{2}\Big((x - y) + (x + y)\Big)$. The possibilities are:

$x - y$	1	3	5
$x + y$	75	25	15
(x, y)	$(38, 37)$	$(14, 11)$	$(10, 5)$

Question 4
The planet Snork is populated by 2 alien species: the Zorks, who always tell the truth, and the Gorks, who always lie. Unfortunately to human eyes the two species look identical.
On the planet you meet 3 aliens: Alt, Balt and Calt. Alt and Balt make the following statements:

Alt: Balt and Calt are the same species.

Balt: At least one of Alt and Calt is a Gork.

Which species is Calt?

Solution: $\boxed{\text{Gork}}$ Denote the aliens by A, B and C. Denote Zorkdom by 1 and Gorkdom by 0.
The statements are: $A:\quad B = C$, $B:\quad$ at least one of $A, C = 0$.
If $A = 1$ then $B = C = 1$ or $B = C = 0$. In the first case $A = B = C = 1$ and B's statement is false, a contradiction. In the second, B's statement is true (since $C = 0$) but this is again a contradiction as $B = 0$.
Thus $A = 0$. So B's statement is true so $B = 1$. But A's statement is false, so $B \neq C$ so $C = 0$, so $(A, B, C) = (0, 1, 0)$.

In the notation of propositional logic we write $\wedge$ for **and**, $\vee$ for **or** and $\neg$ for **not**. We seek an assignment of truth values to A, B, C such that: $A \iff (B = C)$, $\quad B \iff (\neg A \vee \neg C)$. Rewriting these expressions in terms of **and**, **or** and **not**:

$$\begin{array}{rcl} A & \iff & (B = C) \\ (A \wedge (B = C)) & \vee & (\neg A \wedge (B \neq C)) \\ (A \wedge ((B \wedge C) \vee (\neg B \wedge \neg C))) & \vee & (\neg A \wedge ((B \wedge \neg C) \vee (\neg B \wedge C))) \\ (A \wedge B \wedge C) \vee (A \wedge \neg B \wedge \neg C) & \vee & (\neg A \wedge B \wedge \neg C) \vee (\neg A \wedge \neg B \wedge C) \end{array}$$

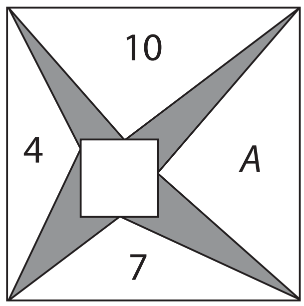
$$\begin{array}{rcl} B & \iff & (\neg A \vee \neg C) \\ (B \wedge (\neg A \vee \neg C)) & \vee & (\neg B \wedge \neg(\neg A \vee \neg C)) \\ (\neg A \wedge B) \vee (B \wedge \neg C) & \vee & (A \wedge \neg B \wedge C) \end{array}$$

Overall:

$$\Big((A \wedge B \wedge C) \vee (A \wedge \neg B \wedge \neg C) \vee (\neg A \wedge B \wedge \neg C) \vee (\neg A \wedge \neg B \wedge C)\Big) \wedge \Big((\neg A \wedge B) \vee (B \wedge \neg C) \vee (A \wedge \neg B \wedge C)\Big)$$

This does simplify using the rules of logic to $(\neg A \wedge B \wedge \neg C)$, which is true exactly if $(A, B, C) = (0, 1, 0)$.

Question 5
A smaller square is placed inside a larger square. The sides of the two squares are parallel. If the areas marked are as shown, what is the area A?



Solution: $\boxed{13}$ Let the distance from the left edge of the large square to the left edge of the small square be x. Let the distance from the top of the large square to the top of the small square be y. Let the large square have side length S, and the small square side length s. Let the areas of the 4 indicated triangles be $A_1 = 10$, $A_2 = 4$, $A_3 = 7$, $A_4 = A$.
Since twice the area of a triangle is the base times the height, we have:

$$\begin{array}{rcl} 2A_1 & = & Sy \\ 2A_2 & = & Sx \\ 2A_3 & = & S(S - s - y) \\ 2A_4 & = & S(S - s - x) \end{array}$$

Thus $2(A_1 + A_3) = S(S - s) = 2(A_2 + A_4)$. So

$$A_4 = A_1 - A_2 + A_3 = 13.$$

For example, $S = 10,\ s = 6.6,\ x = 0.8,\ y = 2$ works, but there are infinitely many possibilities: $(s, x, y) = \left(\frac{S^2-34}{S},\ \frac{8}{S},\ \frac{20}{S}\right)$.