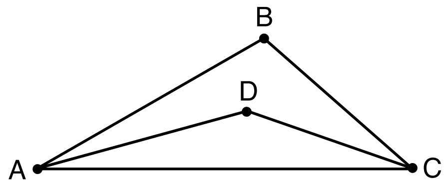
2006 QAMT Problem-Solving Competition - Year 8 Paper

All questions have equal value.

Question 1
The game of Scrabble gives values to letters: $A = 1$, $B = 3$, $C = 3$, $D = 2$, $E = 1$, $F = 4$, $G = 2$, $H = 4$, $I = 1$, $J = 8$, $K = 5$, $L = 1$, $M = 3$, $N = 1$, $O = 1$, $P = 3$, $Q = 10$, $R = 1$, $S = 1$, $T = 1$, $U = 1$,$V = 4$, $W = 4$, $X = 8$, $Y = 4$, $Z = 10$. The value of a word is the sum of the values of the letters. For example, ONE $= 1 + 1 + 1 = 3$. The name of which positive integer has Scrabble score equal to itself?
Solution $\boxed{\text{Twelve}}$ $12 = 1 + 4 + 1 + 1 + 4 + 1$.

Question 2
In triangle ABC, angle ABC is 120°. Point D is chosen in the triangle so that line DA bisects angle BAC and line DC bisects angle BCA. What is the angle ADC?



Solution From triangle ABC $2x + 2y + 120 = 180$, while from triangle ADC $x + y + \alpha = 180$. Multiplying by 2 gives $2x + 2y + 2\alpha = 360$. Combining these gives $2\alpha 120 = 180$, so $\alpha = 150$.

Question 3
Which fraction x with $\frac{1}{10} < x < \frac{1}{9}$ has smallest (positive) denominator?
Solution $\boxed{2/19}$ Suppose $1/10 < a/b < 1/9$ with a, b positive integers. Then $9a < b < 10a$. $a = 1$ is impossible, and $a = 2$ gives $a/b = 2/19$. If $a \geqslant 3$ then $b > 27$.

Question 4
What are the last two decimal digits of 61^{2006}?
Solution $\boxed{61}$ 61^5 ends with last two digits 01, so $61^{2006} = 61 \cdot (61^5)^{401}$ ends with $61 \cdot (01)^{401} = 61$.

Question 5
In front of you is a pile of coins, containing between 1500 and 3000 coins. If you divide it into groups of 7, 11 or 13 each time you have 4 coins left over. How many coins are in the pile?
Solution $\boxed{2006}$ The obvious solution is 4 coin, but this impossible since we have at least 100 coins. If we add any multiple of $7 \cdot 11 \cdot 13 = 1001$, this does not change the remainder upon dividing by 7, 11 or 13, so $2002 + 4$ is a solution. Alternative solution: let n be the number of coins. Then $n = 7x + 4 = 11y + 4$ for some integers x and y so $7x = 11y$ so x must be divisible by 11. Thus $n = 77z + 4$. But also $n = 13w + 4$ for some w, so now z is divisible by 13, and we get $x = 1001t + 4$ for some t. To finish with between 1500 and 3000 coins the only possible t we can take is 2, leading to 2006 coins. It is easy to check that this does indeed give a solution.

Question 6

Suppose $x^3 - x - 1 = 0$. Show that $x - 1 = \dfrac{1}{x^4}$.

Solution First note that $x \neq 0$. Now $x^3 = x + 1$, so $x^4 = x^2 + x$ so $x^5 = x^3 + x^2 = x^2 + x + 1$ so $x^5 - x^4 = 1$. Now divide by x^4.

A real number $x > 1$ *satisfying* $x+1 = x^m$ *and* $x-1 = x^{-n}$ *for some* m *and* n *is called* morphic. *Draw a mark on the real line at positions* $\ldots, x^{-2}, x^{-1}, 1, x, x^2, \ldots$. *This gives a sort of geometric ruler. The morphic condition is that the sum and difference of consecutive intervals coincide with other intervals. By studying polynomials* $x^r - x^s - 1$ *it can be shown that the only morphic numbers are the* $x \simeq 1.3247$ *above and the* golden ratio $(\sqrt{5}+1)/4$.