THE UNIVERSITY OF QUEENSLAND
AUSTRALIA

2005 QAMT Problem-Solving Competition - Year 8 Paper

Question 1 Using the numbers 1 to 9 each once, fill in the boxes to make the equation correct.

$$\square\square \times \square = \square\square \times \square = \square\square \times \square$$

2 marks

Solution
$18 \cdot 9 = 27 \cdot 6 = 54 \cdot 3$.

Question 2 Arrange 6 matches of equal length to form 8 equilateral triangles. The matches must not be broken and must be laid end to end, although they are allowed to cross.

3 marks

Solution
The Star of David is one solution.

Question 3 What are the largest and smallest number of Friday the 13ths that can occur in any year (Jan 1st to Dec 31st)?

4 marks

Solution
February 13th is always $31 = 4 \cdot 7 + 3$ days after January 13th, so is offset by being 3 days later in the week. Extending this calculation we obtain the following offsets:

	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Non-leap	0	3	3	6	1	4	6	2	5	0	3	5
Leap	0	3	4	0	2	5	0	3	6	1	4	6

The number of times each offset occurs is

	0	1	2	3	4	5	6
Non-leap	2	1	1	3	1	1	2
Leap	3	1	1	2	2	1	2

Since $0, 1, \ldots, 6$ all occur at least once in each case, no matter which day January 13th falls on there will always be at least one Friday the 13th. The maximum possible number is 3, which occurs in non-leap years if January 13th is a Tuesday, and in leap years if January 13th is a Friday.

Question 4 According to a survey, at least 70% of people like apples and at least 75% like bananas. What can you say about the percentage of people who like both?

4 marks

Solution
Let a be the percentage who like apples but not bananas, b who like bananas but not apples, and c the percentage who like both. Then we want to maximize/minimize c given constraints:

$$\begin{aligned} a, b, c &\geqslant 0 \\ a + b + c &\leqslant 100 \\ a + c &\geqslant 70 \\ b + c &\geqslant 75 \end{aligned}$$

The second inequality gives $c \leqslant 100 - (a + b) \leqslant 100$ and $c = 100$, $a = b = 0$ is a valid solution. Adding the last two inequalities, $a + b + 2c \geqslant 145$, or $-a - b - 2c \leqslant -145$. Adding the second inequality, $-c \leqslant -45$ or $c \geqslant 45$. And $c = 45$, $a = 25$, $b = 30$ is a valid solution. So

$$45 \leqslant c \leqslant 100.$$

Problems of this type are called *linear programming* problems.

Question 5 Suppose you have four rectangles of size 1×2, three of size 2×3, two of size 3×3, and one of size 5×5. Can you arrange them to form a single large rectangle?

4 marks

Solution
No. The total area is $69 = 3 \cdot 23$, so the only possible dimensions for the large rectangle would be 3×23 or 1×69, and the 5×5 square will not fit in either.

Question 6 Suppose x and y are positive integers satisfying $13x + 8y = 2005$. What is the minimum possible value of $x + y$?

3 marks

Solution
$2005 = 8(x + y) + 5x$ so we should make x as large as possible. But $13x \leqslant 2005 - 8$ and x is an integer, so $x \leqslant 153$. If $x = 153$, $y = 2$ so the smallest $x + y$ can be is 155, with $x = 153$, $y = 2$.