THE UNIVERSITY OF QUEENSLAND
AUSTRALIA

2013 UQ/QAMT Problem Solving Competition - Year 11 & 12 Paper

All questions have equal value. Non-CAS calculators may be used.

Question 1

Several regular polygons meet at a point in the plane leaving no gap. What is the largest number of sides any of the polygons can have?

Solution: $\boxed{42}$ The condition for the polygons to meet with no gap is that the sum of the angles meeting at the point be 360°. For any regular n-gon, the sum of the exterior angles is 360°, so each exterior angle is $360/n$, so each interior angle is $180(1-2/n)$. The first few values are 60, 90, 108, 120, ... for the equilateral triangle, square, pentagon, hexagon, ...

Suppose we have n_1-gon, n_2-gon, ..., n_k-gon meeting snugly at a point. Denote this by $[n_1,\ldots,n_k]$. Thus $180(1-2/n_1+1-2/n_2+\cdots 1-2/n_k)=360$, which gives

$$\frac{k}{2}-1=\frac{1}{n_1}+\frac{1}{n_2}+\cdots+\frac{1}{n_k}.$$

We may **assume** $3\leqslant n_1\leqslant n_2\leqslant\cdots\leqslant n_k$ from now on. Note that as the n_j increase the value of the sum above decreases. Indeed the RHS is $\leqslant k/n_1$. Solving:

$$3\leqslant n_1\leqslant\left\lfloor\frac{2k}{k-2}\right\rfloor$$

We can recursively solve equations of this type by letting n_1 run through the allowable range and solving a similar equation in $k-1$ variables.

Now clearly $k\geqslant 3$ (either from the geometry; all interior angles are less than 180°, or else from putting $k=2$ in the equation). If $k\geqslant 7$ then from above $n_1\leqslant 2$ which is impossible (or else, just observe that 60° is the smallest allowable angle, and 6 copies of it already use up all 360°). Thus $\boxed{k=3,4,5\text{ or }6}$.

$\boxed{k=6}$ $n_1=3$. The only possibility is $[3,3,3,3,3,3]$.

$\boxed{k=5}$ $n_1=3$. $3/2=1/n_1+\cdots+1/n_5$. If $n_4=3$ then $3/2=4/3+1/n_5$ which gives $[3,3,3,3,6]$. (Combine two triangles into a hexagon in the previous case.) If $n_4=4$ then solving gives $[3,3,3,4,4]$. If $n_3=4$ then we need $1/n_4+1/n_5=7/12$ with $n_4,n_5\geqslant 4$, but then $1/n_4+1/n_5\leqslant 2/4<7/12$.

$\boxed{k=4}$ $n_1=3$ or 4. $1=1/n_1+\cdots+1/n_4$. If $n_1=4$ then $1/n_2+1/n_3+1/n_4\geqslant 3/4$, which is only possible with $[4,4,4,4]$. Thus we may now assume $n_1=3$, so $1/n_2+1/n_3+1/n_4=2/3$. Similarly we get solutions $[3,4,4,6]$, $[3,3,6,6]$, $[3,3,4,12]$.

$\boxed{k=3}$ $n_1\leqslant 6$. $\frac{1}{2}=1/n_1+1/n_2+1/n_3$. Similar analysis shows the solutions are $[6,6,6]$, $[5,5,10]$, $[4,8,8]$, $[4,6,12]$, $[4,5,20]$, $[3,12,12]$, $[3,10,15]$, $[3,9,18]$, $[3,8,24]$, $[3,7,42]$.

Note: The equation above is $1/2=1/h+1/k$ where h is the *harmonic mean* of $n_1,\ldots,n_k$. It is well known that the harmonic mean is $\leqslant m$, the (arithmetic) mean. So for example when $k=3$, $m\geqslant h=6$, so the mean number of sides is at least 6.

Question 2

In terms of n, what is the coefficient of x^{n-2} in the expansion of $(x+1)(x+2)...(x+n)$?

Hint: the formula $1^2+2^2+\cdots+k^2=\frac{1}{6}k(k+1)(2k+1)$ *may be useful.*

Solution: $\boxed{(n-1)n(n+1)(3n+2)/24}$ The coefficient is $\sum_{1\leqslant i<j\leqslant n} ij=\frac{1}{2}\left[\left(\sum_{i=1}^n i\right)^2-\sum_{i=1}^n i^2\right]=\frac{1}{8}\left[n^2(n+1)^2-8\sum_{i=1}^n i^2\right]$. The result follows on substituting for $\sum_i i^2$.

Alternative: the coefficient is $\sum_{1\leqslant i<j\leqslant n} ij=\sum_{i=1}^{n-1} i\sum_{j=i+1}^{n} j=\frac{1}{2}\sum_{i=1}^{n-1} i(n-i)(n+i+1)=\frac{1}{2}\Big((n^2+n)\sum_{i=1}^{n-1} i-\sum_{i=1}^{n-1}(i^3+i^2)\Big)=\frac{1}{4}\Big((n-1)n^2(n+1)-2\sum_{i=1}^{n-1}(i^3+i^2)\Big)$. Now we need the formula for the sum of i^2 and i^3. The latter is easy to guess and can be confirmed by induction.

Question 3

A triangle has internal angles α, β and γ at vertices A, B, C respectively. A point P inside the triangle is such that $\angle BAP = \angle CBP = \angle ACP = 30°$. What is the value of

$$\frac{1}{\tan\alpha} + \frac{1}{\tan\beta} + \frac{1}{\tan\gamma}?$$

Solution: $\boxed{\sqrt{3}}$ Let the sides opposite angle α, β, γ have lengths a, b, c respectively. Let $|AP| = x$, $|BP| = y$, $|CP| = z$, and let the area of the triangle be A. Let the desired sum be S.

We find $\sin\alpha$ and $\cos\alpha$ in terms of a, b, c and A. Similar expressions for β, γ by symmetry.

The law of cosines gives

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}.$$

To calculate $\sin\alpha$ we effectively use the law of sines. Recall that a triangle with sides of length r and s meeting at angle θ has area $\frac{1}{2}rs\sin\theta$ (as is easily proved).

Thus $A = \frac{1}{2}bc\sin\alpha$, so

$$\frac{1}{\sin\alpha} = \frac{bc}{2A}.$$

Hence

$$\frac{1}{\tan\alpha} = \frac{b^2 + c^2 - a^2}{4A}.$$

We obtain similar expressions with β, γ, by permuting a, b, c (note that A is the same for each). Adding these together:

$$S = \frac{a^2 + b^2 + c^2}{4A}. \tag{1}$$

Summing the areas of the 3 internal triangles:

$$A = \frac{1}{4}(ay + bz + cx). \tag{2}$$

Using the law of cosines gives $x^2 = b^2 + z^2 - \sqrt{3}bz$, and similar equations for y^2 and z^2. Summing all of these gives

$$ay + bz + cx = \frac{a^2 + b^2 + c^2}{\sqrt{3}}. \tag{3}$$

Substituting equations 3 and 2 into 1 gives the result:

$$\frac{1}{\tan\alpha} + \frac{1}{\tan\beta} + \frac{1}{\tan\gamma} = \sqrt{3}.$$

If we write $\cot\theta$ for $1/\tan\theta$, then we have $\cot\alpha + \cot\beta + \cot\gamma = \cot 30°$.

Question 4

An eggsellent merchant sells eggs. One day a customer walked up and said, "I'll buy half of all your eggs, plus half an egg." The merchant sold the appropriate number of eggs. Soon after that another customer came along and said exactly the same thing, and again bought the appropriate number of eggs. Customers continued arriving and repeating the same request but the merchant lost count after the third customer.

Eventually the eggsellent merchant had sold all his eggs. The merchant only stocks prime quality merchandise, and so he is certain that the number of eggs he had originally was prime. What is the smallest number of customers he could have served?

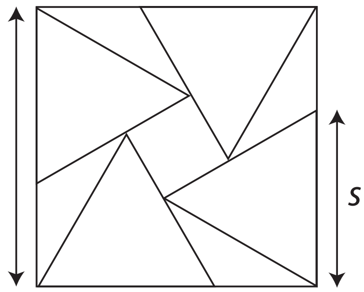
Solution: $\boxed{5}$ Let x_i ("eggxs") be the ("eggxact") number of eggs left after serving i customers, $i = 0, 1, \ldots$. Thus

$$x_{i+1} = x_i - \left(\frac{1}{2}x_i + \frac{1}{2}\right), \qquad x_4 = 0$$

and we need to find x_0. Thus $x_i = 2x_{i+1} + 1$. So $x_3 = 1$, $x_2 = 3$, $x_1 = 7$, $x_0 = 15$.

Note that these are of the form $2^n - 1$. With 4 customers the initial number of eggs would be 15, which is not prime. With 5 customers it would be 31, which is, so 5 customers is the minimum.

Question 5
Four equilateral triangles with side length s are placed in a square of side length 1, as shown. What is the value of s?



Solution: $\boxed{\frac{3-\sqrt{3}}{2}}$