THE UNIVERSITY OF QUEENSLAND
AUSTRALIA

2012 UQ/QAMT Problem Solving Competition - Year 11 & 12 Paper

All questions have equal value.

Question 1
Thirteen pirates are dividing a treasure of approximately 1000 doubloons between them. After dividing the coins into 13 equal heaps, there are 9 coins left over. To avoid an unfair division, 2 of the junior pirates are killed, and the loot redivided into 11 equal heaps. There are again 9 doubloons left over. One more pirate is killed, and now the coins divide exactly between the remaining pirates. Exactly how many coins are in the treasure?

Solution: $\boxed{1010}$ Let x be the number of coins. Then $x-9$ is a multiple of both 11 and 13. Thus $x = 143y + 9$ for some integer y. Since $x = 140y + (3y - 1)$, it follows that $3y - 1$ is divisible by 10. The same is true for $7(3y - 1) = 20y + (y - 7)$, so $y - 7$ must be a multiple of 10, say $y = 10z + 7$. Hence $x = 1430z + 1010$ for some integer z. The only possible choice of z yielding $x \simeq 1000$ is $z = 0$ and $x = 1010$.

Question 2
Suppose $0 < a < c$ and let

$$x = \frac{\sqrt{c+2a} - \sqrt{c+a}}{\sqrt{c} - \sqrt{c-a}}, \qquad y = \frac{\sqrt{c+2a} - \sqrt{c+a}}{\sqrt{c+a} - \sqrt{c}}, \qquad z = \frac{\sqrt{c} - \sqrt{c-a}}{\sqrt{c+2a} - \sqrt{c+a}}.$$

Which of these three numbers, x, y or z, is largest?

Solution: $\boxed{z}$ The condition $0 < a < c$ ensures that all the square roots are real numbers, and x, y and z are all positive. Note also that $z = 1/x$, and that x and y have the same numerator.
The square root function is "concave down" so $\frac{1}{2}\left(\sqrt{c-a} + \sqrt{c+a}\right) < \sqrt{c}$. Algebraically:

$$\begin{aligned}
2c &> 2\sqrt{c^2 - a^2} \\
\therefore\quad 4c &> (c-a) + (c+a) + 2\sqrt{c^2 - a^2} \\
&= \left(\sqrt{c-a} + \sqrt{c+a}\right)^2 \\
\therefore\quad 2\sqrt{c} &> \sqrt{c-a} + \sqrt{c+a} \\
\\
\sqrt{c} - \sqrt{c-a} &> \sqrt{c+a} - \sqrt{c}
\end{aligned}$$

Thus x has larger denominator than y, so $\boxed{x < y}$.
By the same argument with $c+a$ in place of c, the numerator of y is smaller than the denominator, so $1/z = x < y < 1$ so $z > 1 > y$. Thus $x < y < 1 < z$ and z is largest.

Question 3
Bob is playing a two-game chess match. Winning a game scores 2 points, and drawing a game scores 1 point. After the two games are played, the player with more points is declared the champion. If the two players are tied after two games, they continue playing until somebody wins a game (and hence the match).

During each game, Bob can choose to play *boldly* or *conservatively*. If he plays boldly, he has a 45% chance of winning and a 55% chance of losing that game. If he plays conservatively, he has a 90% chance of drawing and a 10% chance of losing. Assuming Bob follows an optimal strategy, what is the probability he wins the match?

Solution: $\boxed{0.566}$ Let $p_j(s)$ be the maximum probability of winning the match if Bob starts game j with s points. We can work backwards to calculate $p_1(0)$, the value we need. If it gets to a tie (game 3) then Bob must play boldly so $p_3(2) = 0.55$. Otherwise $p_3(0) = p_3(1) = 0$ and $p_3(3) = p_3(4) = 1$.
For the first two games we have

$$p_j(s) = \max \begin{cases} 0.45 \times p_{j+1}(s+2) + 0.55 \times p_{j+1}(s) & \text{playing boldly} \\ 0.9 \times p_{j+1}(s+1) + 0.1 \times p_{j+1}(s) & \text{playing conservatively} \end{cases}$$

Thus $p_2(0) = \max\{0.45\times0.55+0.55\times0, 0.9\times0+0.1\times0\} = 0.2475$, $p_2(1) = \max\{0.45\times1+0.55\times0, 0.9\times0.55+0.1\times0\} = 0.495$ and $p_2(2) = \max\{0.45 \times 1 + 0.55 \times 0.55, 0.9 \times 1 + 0.1 \times 0.55\} = 0.955$. Together

$$p_1(0) = \max \begin{cases} 0.45 \times p_2(2) + 0.55 \times p_2(0) & \text{playing boldly} \\ 0.9 \times p_2(1) + 0.1 \times p_2(0) & \text{playing conservatively} \end{cases} = 0.566, \text{playing boldly.}$$

Thus Bob should play boldly on the first game. If he wins he should play conservatively on the second game but if he loses then he should play boldly again on the second game.

Question 4

Three concentric circles of radii 1, 2, 3 respectively are drawn. An equilateral triangle is placed so that each of its vertices lies on a different circle, as shown. What is the side length of the triangle?

Solution: $\boxed{\sqrt{7}}$ Let the side length be s. Join each vertex to the common centre O of the circles. This introduces 3 new triangles with a vertex at O where they form angles α, β and $\alpha+\beta$. Apply the Law of Cosines to each triangle:

$$\begin{aligned} s^2 &= 4+9-12\cos(\alpha) \\ &= 10-6\cos(\beta) \\ &= 1+4-4\cos(\alpha+\beta) \end{aligned}$$

Let $a = \cos(\alpha)$, $b = \cos(\beta)$. Thus $a = (13-s^2)/12$ and $b = (10-s^2)/6$. Now $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta = ab - \sqrt{(1-a^2)(1-b^2)}$. Thus

$$(s^2-5+4ab)^2 = 16(1-a^2)(1-b^2), \qquad a = \frac{13-s^2}{12}, \quad b = \frac{10-s^2}{6}.$$

Substituting in and simplifying we obtain

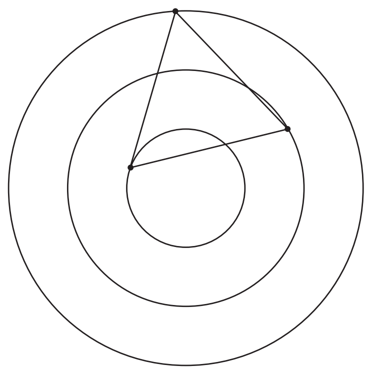
$$s^2(s^2-7)^2 = 0$$

Since $s > 0$, $s = \sqrt{7}$.
Generalization: suppose the 3 radii are r_1, r_2, r_3, with $r_1 < r_2 < r_3$.
The same calculation gives

$$s^2 = \frac{1}{2}\left(r_1^2+r_2^2+r_3^2\right) \pm 2\sqrt{3}\Big(S(S-r_1)(S-r_2)(S-r_3)\Big)^{\frac{1}{2}}, \qquad S = \frac{1}{2}(r_1+r_2+r_3).$$

To have real positive solution we must have $r_1 + r_2 \geqslant r_3$. In this case, let $\triangle$ be the triangle with side lengths r_1, r_2, r_3. Then by Hero(n)'s formula

$$s^2 = \frac{1}{2}\left(r_1^2+r_2^2+r_3^2\right) \pm 2\sqrt{3}\,\text{area}(\triangle).$$



Question 5

Kato the wonder cat has 4 socks and 4 boots and she is puzzled by how many ways she can get dressed. Each sock and boot fits only one paw and each sock must be put on before the corresponding boot. In how many different ways can she put on her socks and boots?

Solution: $\boxed{2520}$