2006 QAMT Problem-Solving Competition - Year 11 & 12 Paper

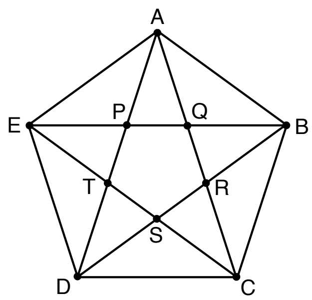
All questions have equal value.

Question 1
Find all x such that

$$\sqrt{\frac{1+\cos 4x}{2}} = \frac{1}{\sqrt{2}}.$$

Solution We need $\cos 4x = 0$, so $4x = k\pi + \pi/2$, so $x = (2k+1)\pi/8$ for any integer k.

Question 2
Connect all the vertices of a regular pentagon ABCDE to each other with straight lines as shown. What is the ratio of the area of ABCDE to the the smaller pentagon PQRST? You may use the fact that $\cos(36°) = \frac{\sqrt{5}+1}{4}$.



Solution The large pentagon has $\boxed{\frac{7+3\sqrt{5}}{2} \simeq 6.85}$ times the area of the smaller.
Let the ratio be ρ. The area of a regular pentagon is proportional to the square of its side length, so the required ratio ρ is $(BC/SR)^2$. Consider triangle RCD. There are various ways to work out its internal angles. For example if O is the centre of the pentagon, then angle RCS is half the angle at the centre ACE, and this is $360°/5$, so angle RCS is $36°$. Similarly angle EBC is $36°$ and this stands on the same arc as ECD, so angle SCD is $36°$ also. A similar argument shows CDR is $36°$, so DRC is $72°$.
Let CD$=x$, RS$=y$, SC$=z$. Triangles SRC and RCD have the same internal angles, hence are similar, so $y/z = z/x$. Thus $\rho = (x/y)^2 = (x/z)^4$. Using the Cosine Rule in triangle SRC, $z^2 = 2x^2(1-\cos 36°) = x^2(\frac{3-\sqrt{5}}{2})$, so $\rho = \frac{4}{(3-\sqrt{5})^2} = \frac{2}{7-3\sqrt{5}} = \frac{2(7+3\sqrt{5})}{4} = \frac{7+3\sqrt{5}}{2}$. This is actually ϕ^4 where $\phi = z/x = (\sqrt{5}+1)/2$ is the *golden ratio.*

Question 3
Let $f = x^3 - px + q$ have roots α, β, γ. Express $(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2$ in terms of p and q.
Solution We can write $f = (x-\alpha)(x-\beta)(x-\gamma)$. Expand this out and equate coefficients. This gives

$$(1) \qquad \alpha+\beta+\gamma = 0, \quad \alpha\beta+\beta\gamma+\gamma\alpha = -p, \quad \alpha\beta\gamma = -q.$$

Thus $\gamma = -\alpha-\beta$ and $p = -\alpha\beta - \gamma(\alpha+\beta) = -\alpha\beta + (\alpha+\beta)^2 = \alpha^2+\alpha\beta+\beta^2 = (\alpha-\beta)^2 + 3\alpha\beta$. Thus

$$(\alpha-\beta)^2 = p - 3\alpha\beta.$$

By symmetry, we get similar expressions for $\beta-\gamma$ and $\gamma-\alpha$, so

$$\Delta = (p-3\alpha\beta)(p-3\beta\gamma)(p-3\gamma\alpha).$$

So

$$\Delta = p^3 - 3(\alpha\beta+\beta\gamma+\gamma\alpha)p^2 + 9\alpha\beta\gamma(\alpha+\beta+\gamma) - 27(\alpha\beta\gamma)^2.$$

Using Equation 1 we get $\boxed{\Delta = 4p^3 - 27q^2.}$

This is called the discriminant *of the cubic polynomial, and plays the same role as* b^2-4ac *for quadratic polynomials.*

Question 4

The *Koch snowflake* is constructed as follows. Start with an equilateral triangle of side length 1. Now divide each side into 3 equal parts and construct an equilateral triangle extending outwards on the middle section, and then delete its base. Repeatedly apply this subdivision step. The snowflake is the limiting shape obtained. What is its area?

Solution The area of the triangles added at each stage is 1/9 times the area of the triangles at the previous stage, and the number of triangles added at stage n is $3\cdot 4^{n-1}$. The initial triangle has area $\sqrt{3}/4$, so the total area is $\sqrt{3}/4+\sqrt{3}/4(1/3+4/3^3+4^2/3^5+\cdots) = 2\sqrt{3}/5$.

Question 5

You have 6 coins, labelled 1 to 6 on one side and blank on the other. You toss all the coins into the air, let them fall, then make your best guess as to which coin is labelled 1. What is the probability you are correct?

Solution There is a 1/2 chance that number 1 comes upright. Suppose it does not. Out of the 5 remaining coins, there is a 1/32 chance that all fall face up. In this case we can identify coin 1 with certainty. There are 5 ways that exactly 4 coins can fall face up, and then we have a 1/2 chance of guessing coin 1. There are $5\cdot 4/2$ ways that exactly 3 coins can fall face up, leaving 2 face down (plus coin 1) so we have a 1/3 chance in this case. Etc. Altogether the probability is

$$\frac{1}{2}+\frac{1}{2}\left(\frac{1}{32}+\frac{5}{32}\cdot\frac{1}{2}+\frac{10}{32}\cdot\frac{1}{3}+\frac{10}{32}\cdot\frac{1}{4}+\frac{5}{32}\cdot\frac{1}{5}+\frac{1}{32}\cdot\frac{1}{6}\right)=\frac{85}{128}.$$

The first 1/2 is the chance that number 1 comes upright. The 1/2 factor in front of the parentheses is the probability that number 1 is face down.

The total is $\boxed{\frac{85}{128}}$.

Question 6

The hyperbolic functions are defined for all real numbers x as follows:

$$\sinh x = \frac{e^x-e^{-x}}{2}, \qquad \cosh x = \frac{e^x+e^{-x}}{2}, \qquad \tanh x = \frac{\sinh x}{\cosh x}.$$

Show that $\arcsin(\tanh x) = \arctan(\sinh x)$.

Solution We first make some remarks about the domains of the various functions. $\sin:[-\pi/2,\pi/2]\to[-1,1]$, so $\arcsin:[-1,1]\to[-\pi/2,\pi/2]$ while $\tan:(-\pi/2,\pi/2)\to\mathbb{R}$ so $\arctan:\mathbb{R}\to(-\pi/2,\pi/2)$. Note that $\cosh x>0$ for all x, and $-e^x-e^{-x}<e^x-e^{-x}<e^x+e^{-x}$. Thus $-1<\tanh x<1$ for all x, so $\tanh:\mathbb{R}\to(-1,1)$, while $\sinh:\mathbb{R}\to\mathbb{R}$. Note also that $e^{2x}\geqslant 1$ for $x\geqslant 0$ so $e^x\geqslant e^{-x}$ so $\sinh x$ and hence $\tanh x\geqslant 0$ for $x\geqslant 0$.

Let $y=\arcsin(\tanh x)$. Since $-1<\tanh x<1$, y is defined and $\sin y=\tanh x$. We need to show $\tan y=\sinh x$. Now $\tan^2 y=\frac{\sin^2 y}{1-\sin^2 y}=\frac{\tanh^2 x}{1-\tanh^2 x}=\frac{\sinh^2 x}{\cosh^2 x-\sinh^2 x}$. But $\cosh^2 x-\sinh^x = ((e^x+e^{-x})^2-(e^x-e^{-x})^2)/4 = (e^{2x}+2+e^{-2x}-e^{2x}+2-e^{-2x})/4=1$. So $\tan y=\pm\sinh x$.

If $x\geqslant 0$ then $\tanh x\geqslant 0$ so $y\geqslant 0$ so $\tan y\geqslant 0$ and $\sinh x\geqslant 0$ so the sign must be $+$ in this case. Similarly for $x<0$.