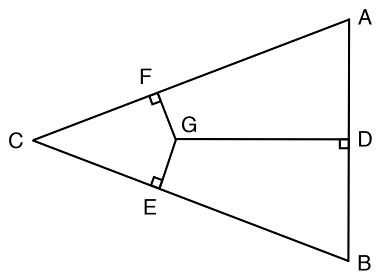
2005 QAMT Problem-Solving Competition - Year 11 & 12 Paper

Question 1 If x is a real number, $|x|$ denotes the distance of x from 0. For example $|-3| = 3$, $|5| = 5$. Let R be the set of all points (x, y) in the plane satisfying $|x| - |y| \leqslant 1$ and $|y| \leqslant 1$. Find the area of R.

2 marks

Solution
The region is bounded between the lines $y = \pm 1$. If $x, y \geqslant 0$ then $x - y \leqslant 1$ so $y \geqslant x - 1$. If $x \geqslant 0$ but $y \leqslant 0$ then $y \leqslant 1 - x$ and so on. We end up with the rectangle with $-1 \leqslant y \leqslant y$ and $-2 \leqslant x \leqslant 2$, minus 2 triangles of height 1 and base 2, so the total area is 6. See diagram.

Question 2 In the diagram below, $FG + EG = DG$, $EG + DG = DA = 2EC = AF - FG$. Find the ratio FC/EG.



4 marks

Solution
Join G to A and C. Let $EC = a$, $EG = b$, $FG = c$, and $FC = d$. ¿From the second given equation, $AD = 2a$, $AF = 2a + c$, $DG = 2a - b$, and from the first $c = 2a - 2b$, so $AF = 4a - 2b$. Note that $c = 2(a - b) \geqslant 0$ so $a \geqslant b$. We need to find d/b, which we denote by y.
Use Pythagoras on the lengths AG, CG:

$$\begin{aligned} d^2 &= a^2 + b^2 - (2a - 2b)^2 \\ (4a - 2b)^2 + (2a - 2b)^2 &= (2a - b)^2 + (2a)^2 \end{aligned}$$

Let $x = a/b$. Note that $x \geqslant 1$. Divide through by b^2. Note that

$$\begin{aligned} y^2 &= x^2 + 1 - 4(x - 1)^2 = 8x - 3x^2 - 3 \\ 4(2x - 1)^2 + 4(x - 1)^2 &= (2x - 1)^2 + 4x^2 \end{aligned}$$

The last equation gives $0 = 12x^2 - 20x + 7 = (2x - 1)(6x - 7)$ so $x = 1/2$ or $7/6$, but $x \geqslant 1$ so $x = 7/6$. Substituting, $y^2 = 9/4$ so $y = 3/2$.

Question 3 Five letters are written to five different addresses, and five matching envelopes with the addresses are prepared. How many ways can the letters be placed into the envelopes (one in each envelope) such that every letter is placed in a wrong envelope?

4 marks

Solution

Inclusion/Exclusion. There are $5! = 120$ possible arrangements, but $4!$ have the first letter in the first envelope, another $4!$ have the second letter correctly addressed and so on. So we have to subtract $5 \cdot 4!$: all the arrangements that have at least one letter correct. But now we have subtracted too much, because we've subtracted any arrangement with both the first and second letters correct—or any two letters correct—twice. There are $\binom{5}{2}3!$ arrangements with two letters correct, which we have to subtract. But then we have to add back on any arrangement with three letters correctly addressed, of which there are $\binom{5}{3}2!$. And so on. The total number is thus

$$1 \cdot 5! - \binom{5}{1}4! + \binom{5}{2}3! + \binom{5}{3}2! + \binom{5}{4}1! - 1 \cdot 0! = 5! - 5 \cdot 4! + 10 \cdot 3! - 10 \cdot 2! + 5 \cdot 1! - 1 = 44.$$

Question 4 Alice, Bob and Cathy take turns (in that order) in rolling a six sided die. If Alice ever rolls a 1, 2 or 3 she wins. If Bob rolls a 4 or a 5 he wins, and Cathy wins if she rolls a 6. What is the probability that Cathy wins?

3 marks

Solution
Let the probability that Cathy wins be p. The probability Alice wins on her first roll is $1/2$. The probability Bob wins on his first roll is $1/2 \cdot 1/3 = 1/6$ (there is only a $1/2$ chance Alice has not won, so that he gets to roll at all). The probability Cathy wins on her first roll is $(1-1/2-1/6) \cdot 1/6 = 1/18$. The probability that no one wins in the first three rolls is $\frac{3}{6} \cdot \frac{4}{6} \cdot \frac{5}{6}$. After that the game effectively starts again, so

$$p = \frac{1}{18} + \frac{5}{18}p$$

which gives $p = 1/13$.
Alternatively, p is given by

$$p = \frac{1}{18}\left(1 + \frac{5}{18} + \left(\frac{5}{18}\right)^2 + \cdots\right) = \frac{1}{18} \cdot \frac{1}{(1-\frac{5}{18})} = \frac{1}{13}.$$

Question 5 For which integer values of x is $x^3 + 82x^2 - 34x - 2005$ a prime number?

3 marks

Solution
Let $f(x) = x^3+82x^2-34x-2005$. Since $2005 = 5 \cdot 401$ (and 401 is prime), if f factors algebraically, its only possible roots are ± 1, ± 5, ± 401, ± 2005. Testing these we find $f(5) = 0$, so $(x-5)$ is a factor of $f(x)$. Doing long division we find

$$f(x) = (x-5)(x^2+87x+401).$$

Thus if the value $f(x)$ is prime one of these two factors must be ± 1. That is, $x - 5 = \pm 1$ or $x^2 + 87x + 401 = \pm 1$. Testing we find that the two quadratics are not solvable with x an integer (the discriminant is not a square). So $x - 5 = \pm 1$, so $x = 4$ or $x = 6$.
Finally $f(4) = -765 = -3^2 \cdot 5 \cdot 17$ is not prime and $f(6) = 959 = 7 \cdot 137$ is not prime either, so $f(x)$ *never* takes on prime values for integer values of x.

Question 6 Let x, y and z be positive real numbers. Show that

$$(x+y+z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \geqslant 9.$$

4 marks

Solution

Multiplying out, the LHS is

$$3+\left(\frac{y}{x}+\frac{x}{y}\right)+\left(\frac{z}{x}+\frac{x}{z}\right)+\left(\frac{y}{z}+\frac{z}{y}\right).$$

Now

$$\frac{y}{x}+\frac{x}{y}=\frac{x^2+y^2}{xy}=\frac{(x-y)^2+2xy}{xy}\geqslant 2$$

and similarly for the other two pairs of fractions grouped above, so the LHS is $\geqslant 3+2+2+2=9$.