2004 QAMT Problem-Solving Competition - Year 11 & 12 Solutions

Question 1 An integer $n > 1$ is said to be *abundant* if the sum of its factors (including 1 but excluding n) exceeds n. For example 12 is abundant since $1 + 2 + 3 + 4 + 6 = 16 > 12$. Find an odd abundant number less than 1000. $\boxed{\text{2 marks}}$

Solution $945 = 3^3 \cdot 5 \cdot 7$ is the only odd abundant number under 1000. This can be found by trial and error. Below we show that in fact this is the smallest abundant number.
Let $\sigma(n)$ be the sum of all divisors of n (including n itself). We need $\sigma(n)/n > 2$. If $n = p^a$ then $\sigma(n) = 1+p+\cdots+p^a = (p^{a+1}-1)/(p-1)$, so $\sigma(n)/n = (1+\frac{1}{p-1})(1-\frac{1}{p^a}) < 1+1/(p-1) \leqslant 2$. If $n = p^a q^b$ then $\sigma(n) = (1 + p + \cdots + p^a)(1 + q + \cdots + q^b)$, so $p, q \geqslant 3 \implies \sigma(n)/n < (1+1/(p-1))(1+1/(q-1)) \leqslant (1+1/2)(1+1/4) = 15/8 < 2$. Thus n must have at least 3 prime factors, and since $3 \cdot 5 \cdot 7 \cdot 11 = 105 \cdot 11 > 1000$, n has exactly 3 prime factors. If the smallest prime factor is not 3 then $\sigma(n)/n < (1 + 1/4)(1 + 1/6)(1 + 1/10) = 77/48 < 2$, contradiction. Similarly, the next smallest factor is 5. Let the remaining factor be q, so $n = 3^a 5^b q^c$ with $q \geqslant 7$. If $a = 1$ then $\sigma(n)/n < 4/3 \cdot 5/4 \cdot q/(q-1) < 2$ unless $q < 6$ a contradiction. If $a = 4$ then $n \geqslant 81 \cdot 5 \cdot 7 > 1000$, so $a = 2$ or $a = 3$. If b or $c \geqslant 2$ then $n \geqslant 3^2 \cdot 5^2 \cdot 7 > 1000$, so $b = c = 1$. Thus $n = 3^2 \cdot 5 \cdot 7 = 315$, $n = 3^2 \cdot 5 \cdot 11 = 495$, $n = 3^2 \cdot 5 \cdot 13 = 585$, $n = 3^2 \cdot 5 \cdot 17 = 765$ or $n = 3^3 \cdot 5 \cdot 7 = 945$ are the only possibilities. Only 945 is abundant. Thus it is the smallest odd abundant number.

Question 2 Three spherical melons, radius 9cm each, are placed on a flat table, each touching both the others. What is the radius of the largest orange that will sit on the table in the space between the 3 melons? $\boxed{\text{4 marks}}$

Solution 3 cm.
Let the unknown radius be a. Join the centres of the melons and the orange. This gives a tetrahedron $ABCD$. Its base is an equilateral triangle ABC of side length 18cm. Its sloping slides AD, BD, CD are length $9 + a$. Drop a perpendicular from the apex D to triangle ABC, meeting the triangle at Q. Let $DQ = h$, the height of the tetrahedron. Let $QA = m$.
Next show that $m = 6\sqrt{3}\text{cm}$, either by geometry knowledge or using $30 - 60 - 90$ triangles.
Finally $h + a = 9$ and $h^2 + m^2 = (9 + a)^2$, so $a = 3\text{cm}$.

Question 3 Three planets are orbiting a star in circular orbits. The first takes 12 years to make an entire orbit, the second 25 years and the third 32 years. Currently all three planets and the star are aligned along a straight line. How many years pass until this next happens? (The planets do not have to return to their starting positions, they just need to all line up.) $\boxed{\text{4 marks}}$

Solution 2400 years.
The first planet travels 2π radians in 12 years, at an angular velocity of $2\pi/12$ radians per year. After t years it is at angle $t \cdot 2\pi/12$. So the condition for the first two planets to line up is $t \cdot 2\pi/12 = 2k\pi + t \cdot 2\pi/25$, where k is an integer indicating the number of "extra laps" the faster planet has done. Solving, $t = 300k/13$ for some integer k. Similarly planets 1 and 3 align at $t = 96m/5$, and 2 and 3 when $t = 800n/7$. Multiplying, $455t = 2^2 \cdot 3^2 \cdot 5^3 \cdot 17k = 2^5 \cdot 3 \cdot 7 \cdot 13m = 2^5 \cdot 5^3 \cdot 13n$. The lcm is $2^5 \cdot 3 \cdot 5^3 \cdot 7 \cdot 13 = 1092000$, so $t = 1092000/455 = 2400$ years.
2400 is actually the lcm of the orbits. This means they only line up again when they are back in their starting positions!

Question 4 A taxi driver drives from the corner of 1st street and A avenue, to the corner of 5th street and E avenue in the diagram below. Each block counts as 1 unit, so the shortest possible path is 8 units. How many different paths are there of this length? $\boxed{\text{2 marks}}$

Solution The route is described by a sequence of 8 N's or E's (standing for North or East), with exactly 4 N's. The number of such routes is thus $\binom{8}{4} = 70$.

Question 5 Show that $x^4 + y^4 = z^4 + 3$ has no integer solutions for x, y, and z. 4 marks

Solution Any solution is also a solution mod 10 (ie just considering the last decimal digit). But $x^4 \equiv 0, 1, 5, 6 \pmod{10}$, so the lhs is $0, 1, 2, 5, 6, 7$ while the rhs is $3, 4, 8, 9$.

Question 6 Let $S_1 = 1$, $S_2 = 1 + 2$, $S_3 = 1 + 2 + 3$, etc. Evaluate

$$\frac{1}{S_1} + \frac{1}{S_2} + \cdots + \frac{1}{S_{2004}}.$$

4 marks

Solution $1/S_n = 2/n(n+1) = 2(1/n - 1/(n+1))$ so the sum is $2\sum_{n=1}^{2004} 1/n - 1/(n+1) = 2(1 - 1/2004) = 4008/2005$. (Telescoping sum.)