# 2003 QAMT Competition

# Year 11&12 SOLUTIONS

**Question 1.** A credit card number (with 14 digits) is written in the boxes below. If the sum of any three consecutive digits is 20, then what is $y$?

1 mark

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | 9 | | | | $y$ | | | | 7 | | |

**A** 3 **B** 4 **C** 5 **D** 7 **E** 9

**Answer**: B, 4

**Solution**: The digits must be in a cycle of length 3: the first digit matches with the fourth digit: if x1+x2+x3=20, then x2+x3+x4=20 and so x1=x4. Thus,

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| A | B | C | 9 | B | C | A | $y$ | C | A | B | 7 | A | B |

and so A=9, C=7 and $y$=B = 4 = 20=7-9.

**Question 2.** If numbers $a,b,c$ satisfy $a+b+c = 0$ and $a^2+b^2+c^2 = 1$ then what is $a^4+b^4+c^4$?

1 mark

**A** 1/4 **B** 1/2 **C** 1 **D** 4 **E** None of these

**Answer**: B, ½

**Solution**: Can check with some sample values to confirm the answer is B. Note that $a^2+b^2+c^2=1$ means $a,b,c$ must lie between –1 and +1.

Given, $a+b+c=0, \quad a^2+b^2+c^2=1$ so

$$\left(a^2+b^2+c^2\right)^2=1$$
$$\Rightarrow a^4+b^4+c^4+2a^2b^2+2b^2c^2+2a^2c^2=1$$
$$\Rightarrow a^4+b^4+c^4=1-2\left(a^2b^2+b^2c^2+a^2c^2\right)$$

so the result (B) will hold if we can show that $\left(a^2b^2+b^2c^2+a^2c^2\right)=\frac{1}{4}$ (*).

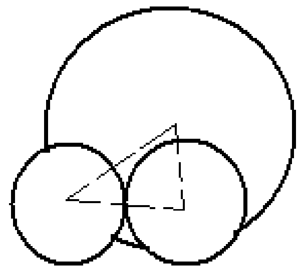
Now, $a+b+c=0$, so $(a+b+c)^2=0$. Hence

$$a^2+b^2+c^2+2ab+2bc+2ac=0$$
$$\Rightarrow 1+2(ab+bc+ac)=0$$
$$\Rightarrow ab+bc+ac=-\frac{1}{2}$$
$$\Rightarrow (ab+bc+ac)^2=\frac{1}{4}$$
$$\Rightarrow (ab)^2+(bc)^2+(ac)^2+2ab^2c+2a^2bc+2bac^2=\frac{1}{4}$$
$$\Rightarrow a^2b^2+b^2c^2+a^2c^2+2abc(a+b+c)=\frac{1}{4}$$
$$\Rightarrow a^2b^2+b^2c^2+a^2c^2=\frac{1}{4}$$

confirming (*) and so the result (B) holds.

1 mark

**Question 3.** Three spheres are just touching each other. The radii of the spheres are 1, 1, and 2 units. What is the area of the triangle joining the centres of the spheres?



**A** π **B** 1/2 **C** 2/√2 **D** 2√2 **E** None of these

**Answer: D, 2**√ 2

**Solution:** Because the spheres are touching, the sides of the triangle must be 2, 3 and 3. The area of the triangle is then: 2√ 2

2 marks

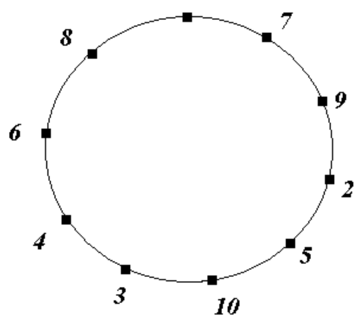
**Question 4**: The whole numbers 1 up to 11 are arranged in any order at *eleven* places on the circumference of a circle. Show that there must be three numbers next to each other in the arrangement whose sum is at least 19.

**Solution**: Suppose we try to arrange the numbers 1..11 so as to avoid a run of three with sum 19 or more. Consider the positions within two places of 11, so ___, ___, 11, ___, ___. If any of these is filled by 10, 9, 8 or 7 then there is a run of three with sum at least 19. (Note that for 7, 11+7 = 18, but the third number in the run is at least 1.) So all of 10, 9, 8, 7 must be in the remaining six places around the circle. The 10 must be separated from both the 9 and 8 by at least two places, or we again have a run of three with sum at least 19; and the 9 and 8 cannot be adjacent
(for ___, 9,8,___ gives three with a sum of at least 19 since one "___" must be at least 2). So we must have 10, ___, ___, 8, ___, 9 or 10, __, ___9, ___, 8. We must use the 1 betweeen the 8 and 9, so that the 7 fills one "___" within two places of the 10. The other "___" is at least 2, and 10+7+2=19. So you cannot avoid a run of three with sum at least 19.

Another method is to consider the total sum of the eleven different groups of three consecutive numbers around the circle. Each number occurs in three groups, so the total sum is $3 \times (1+2+\ldots+10+11)=198$. However, if no group of three consecutive numbers had a sum of 19 or more, the total sum would be at most $11 \times 18 = 198$. This means that to avoid a sum of 19 or more, every run of three numbers must have a sum of 18. But this cannot happen. If a,b,c,d are four adjacent numbers, then $a \neq b$ and so $a+b+c \neq b+c+d$.

In some versions of the competition paper, this question mistakenly asked for

> "…numbers 1 up to 11 to be arranged in any order in ten places on a circle."




If you try the question with this specification, then you can in fact find a counterexample. If you want to find an arrangement where the sum of any three adjacent numbers is less than 19, then, of course, you would select 1,2,…,10. Here is an arrangement where the sum of any three adjacent numbers is less than 19.

**Question 5**: Find all integer solutions to $m^2 + 2mn + 2m + n = m^2n + 1$.

2 marks

**Answer**: (m, n) = (-1, -1), (0, 1), (1, -1), (2, -7), or (3, 7).

**Solution**: We have $m^2(n - 1) = 2m(n + 1) + n - 1 = 2m(n - 1) + 4m + (n - 1)$. Put $k = n - 1$ and this becomes $m^2k = 2mk + 4m + k$ (*).

If $k = 0$, then $m = 0$. This gives the solution $m = 0$, $n = 1$. If $m = 0$, then $k = 0$, so we assume m and k are non-zero. From (*) m must divide k and k must divide 4m. Put $k = mh$. Then h divides 4, so $h = \pm1, \pm2$ or $\pm4$. Also we can write (*) as $(m^2 - 2m - 1)h - 4 = 0$.

If $h = 1$, then $m^2 - 2m - 5 = 0$, which has no integral solutions. If $h = -1$, then $m^2 - 2m + 3 = 0$, which has no integral solutions. If $h = 2$, then $m^2 - 2m - 3 = 0$, which has solutions $m = -1$ or 3. If $h = -2$, then $m^2 - 2m + 1 = 0$, so $m = 1$. If $h = 4$, then $m^2 - 2m - 2 = 0$, which has no integral solutions. If $h = -4$, then $m^2 - 2m = 0$, which has solutions $m = 0, 2$.

**Question 6**: Consider a rectangle with its vertices all on the boundary of a given triangle *T*. Let *d* be the shortest diagonal for any such rectangle. Find the maximum value of:

3 marks

$$\frac{d^2}{Area\ T} \text{ over all triangles } T.$$

**Answer**: $(4\sqrt{3})/7$.

**Solution**: Suppose two vertices are on the side length a and that the corresponding altitude is length h. Let the side parallel to the altitude have length x. Then the other side has length $(h - x)a/h$. So the square of the diagonal is $x^2 + a^2(1 - x/h)^2 = (1 + a^2/h^2)(x - a^2h/(a^2+h^2))^2 + a^2h^2/(a^2+h^2)$. Hence $d^2 = a^2h^2/(a^2+h^2)$, or rather that is true provided we have picked the right side.

Suppose b is another side length with corresponding altitude k and. Let A = area T, then $ah = bk = 2A$. So we require $a^2 + h^2 >= b^2 + k^2$ or $a^2 + 4A^2/a^2 >= b^2 + 4A^2/b^2$ or $(a^2 - b^2)(1 - 4A^2/(a^2b^2)) >= 0$. But $ab >= 2A$ (with equality iff the sides a and b are perpendicular, but the longest side is never perpendicular to another side). So we get the shortest diagonal by taking the longest side.

The required ratio $d^2/A$ is thus $2ah/(a^2 + h^2)$, where a is the longest side. We may assume $A = 1/2$ (since the ratio is unaffected by the scale of the triangles). So we have to maximise $2/(a^2 + h^2)$ or equivalently to minimise $a^2 + 1/a^2$. The area of an equilateral triangle with side 1 is $(\sqrt{3})/4 < 1/2$, so $a > 1$, but $a^2 + 1/a^2$ is an increasing function of a for $a > 1$, so we want to minimise a subject to the constraint that A is fixed. That obviously occurs for an equilateral triangle. An equilateral triangle with side 1 has altitude $(\sqrt{3})/2$, so $d^2/A$ has maximum value $\sqrt{3}/(1 + 3/4) = (4\sqrt{3})/7$.