2002 QAMT Competition
Year 11&12 Paper

Solutions

Q1. (1 point) The last two digits of 2^{222} are:

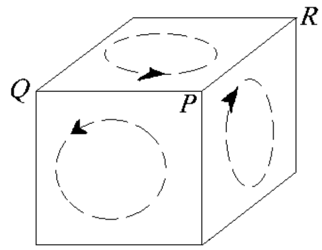
(A) 84, (B) 24, (C) 64, (D) 04, (E) 44

Solution: If you look for a pattern, you will find that the last two digits follow the sequence. The top line is the exponent and the bottom line is the final two digits.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
02	04	08	16	32	64	28	56	12	24	48	96	92	84	68	36	72	44	88	76	52	04

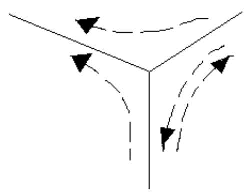
It appears to be a cycle, which starts at exponent 2, and repeats starting at exponent 22. So there seem to be 20 different 2-digit endings. You will find that 221/20 = 11 with a remainder of 1, so the ending for 2^{222} will be the same as for 2^2, which is 04, answer (D).

Q2. (1 point) One ant is placed on each face of a suspended cube. Each ant walks a circuit around the edges of its own face. An edge of the cube is said to be *contested* if the two ants using it do so in opposite directions. Thus, in the diagram, PQ is contested, but PR is not. What is the least possible number of contested edges?



(A) 2, (B) 3, (C) 4, (D) 5, (E) 6

Solution: Consider the corners of the cube. There are two types of vertex in any configuration of paths – there is either an uncontested edge and hence there must then be two uncontested edges at the vertex or all three edges are contested



Suppose that in some configuration of paths there are r vertices where all three edges are contested. Then the number of contested edges on the cube is

½ [3r + (8 - r)] = 4 - $r \geq 4$

The ½ factor is there because each edge is contested twice.

The minimum will occur when r = 0, that is, every vertex has two uncontested edges. This configuration is possible – try it arranging the paths so that paths on opposite faces are clockwise and the other four are anticlockwise. Hence the answer is (C).

Q3. (2 points) Mixture I consists of lemon juice, oil and vinegar in the proportions 1:2:3, while mixture II has them in proportions 3:4:5. Which of the following proportions lemon juice: oil: vinegar is obtainable by combining mixtures I and II?

(A) 2:5:8, (B) 4:5:6, (C) 3:5:7, (D) 5:6:7, (E) 7:9:11

Solution: Note that the proportions in mixtures I and II can be expressed as $\frac{2}{12}:\frac{4}{12}:\frac{6}{12}$ and $\frac{3}{12}:\frac{4}{12}:\frac{5}{12}$ respectively.

Hence in any mixture of the two, $\frac{2}{12} \le$ proportion of lemon juice $\le \frac{3}{12}$.

Considering the proportion of lemon juice in the options,

Answer (A) proportion = 2/15 < 2/12, hence it is not possible

Answer (B) proportion = 4/15 >3/12, hence it is not possible

Answer (D) proportion = 5/18 > 3/12, hence it is not possible

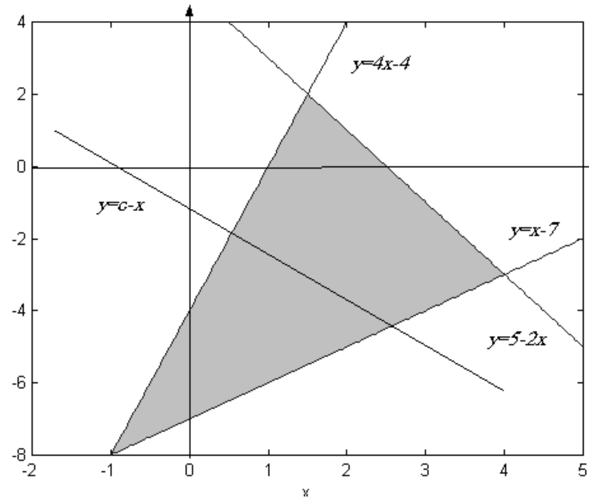
Answer (E) proportion = 7/27 > 3/12, hence it is not possible.

Hence the answer must be (C).

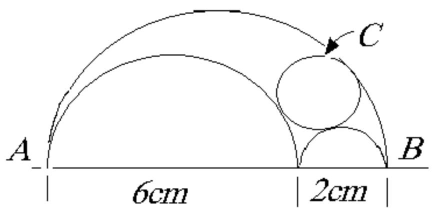
Q4. (2 points) For which values of the constant c does there exist a pair of real numbers (x, y) which satisfies all four conditions:

$$2x + y < 5;\ x - y < 7;\ 4x - y > 4 \text{ and } x + y = c\ ?$$

Solution: The easiest way is to note that the three inequalities hold just for (x,y) in the interior of the triangle ABC shown. Clearly $y = c - x$ meets the interior if and only if it cuts the line $y = 4x - 4$ between A = ($\frac{3}{2}$, 2) and B = (-1,-8), and thus provided $-9 < c < \frac{7}{2}$.



Q5. (2 points) You have a semicircle with diameter AB of length 8cm, inside which are drawn non-overlapping semicircles of diameters 6cm and 2cm. A circle C is drawn, touching all three semicircles as shown. Show that the perpendicular distance from the centre of this circle C to the dimeter AB is equal to the diameter of the circle C.



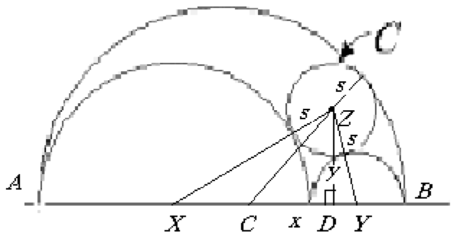
Solution: Let C, X, Y, Z be the centres as shown and D the foot of the perpendicular from AB through Z. Let CD = x cm, DZ = y cm and let the circle C have radius s cm. Then XZ = 3+s, CZ = 4-s and YZ = 1+s.

From ΔCZD, $x^2 + y^2 = (4-s)^2$ (1)

From ΔXZD, $(1+x)^2 + y^2 = (3+s)^2$ (2)

From ΔZYD, $(3-x)^2 + y^2 = (1+s)^2$ (3)

Hence by (1) and (2), $1+2x + (4-s)^2 = (3+s)^2$, so $2x - 14s = -8$ and by (1) and (3),
$9 - 6x + (4-s)^2 = (1+s)^2$, so $-6x - 10s = -24$. Hence $s = 12/13$ and $x = 7s - 4$. Then from (1),
$y^2 = (4-s)^2 - (7s-14)^2 = -6s(-8+8s) = \frac{6 \cdot 12 \cdot 8 \cdot 1}{13 \cdot 13}$, so $y = 24/13$. Thus $y = 2s$ as required.



Q6. (2 points) Find all positive integers *x, y, z* where $x \le y \le z$ and

$$\frac{1}{xy} + \frac{1}{yz} + \frac{1}{zx} = \frac{1}{3}$$

Solution: If $x \le y \le z$ then

$$\frac{1}{xy} \le \frac{1}{yz} \le \frac{1}{zx} \quad \text{so that} \quad \frac{1}{xy} + \frac{1}{yz} + \frac{1}{zx} \le \frac{3}{xy}$$

Hence if $3/xy < 1/3$, no solution is possible, that is if $xy>9$ no solution is possible.

Since $x \le y$ and x is a positive integer this means that x = 1, 2, 3.

Case 1, x=1, so $1 \le y \le 9$. The equation becomes:

$\frac{1}{y} + \frac{1}{yz} + \frac{1}{z} = \frac{1}{3}$ so $3(z+1+y)=yz$ and hence $z = 3(1+y)/(y-3)$. Thus in fact $y \ge 4$ since z is positive. We can draw up the following table of values for y and z:

y	4	5	6	7	8	9
x	15	9	7	6	27/5	5

Here the last three must be discarded since we require $y \le z$ (and z an integer).

Case 2, x=2, so $2 \le y \le 4$. The equation becomes $\frac{1}{2y} + \frac{1}{yz} + \frac{1}{2z} = \frac{1}{3}$,

so $3(z+2+y)=2yz$ and hence $z = 3(2+y)/(2y-3)$. We have the table

y	2	3	4
x	7	5	18/5

The last must be discarded since z must be an integer.

Case 3, x=3. Then y=3 and z=3.

From all three cases, the solution for (x,y,z) are: (1,4,15), (1,5,9), (1,6,7), (2,2,12), (2,3,5), (3,3,3)