Evaluating Persistence Times in Populations that are Subject to Catastrophes

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Intro 1: Queues vs. Populations

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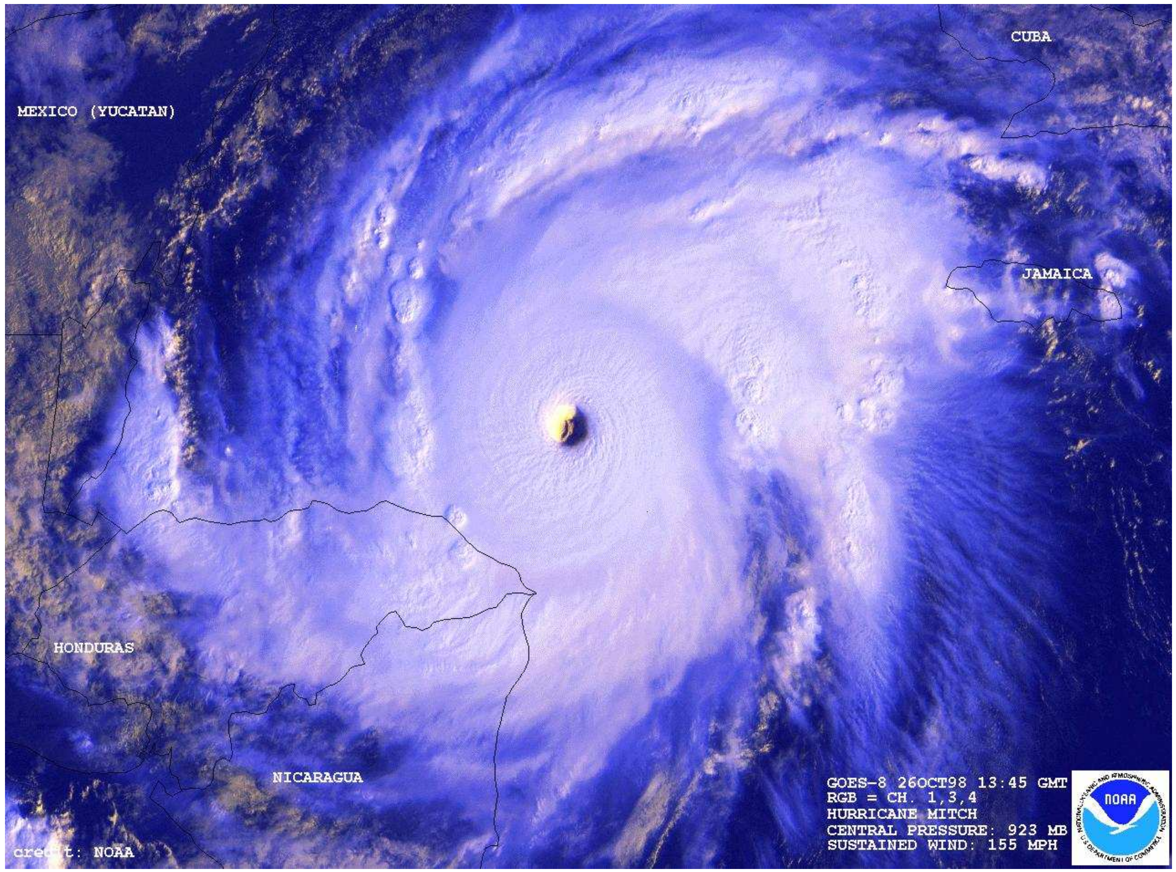
- Of course, any biological population experiences **births** and **deaths**.
  - (c.f. *birth-death processes*.)
- Some populations may also experience **catastrophic** events, which may cause large numbers of deaths.

Introduction: Catastrophes



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We will define *continuous-time Markov chains* exhibiting transitions corresponding to each of these events:

birth, death and catastrophe processes.

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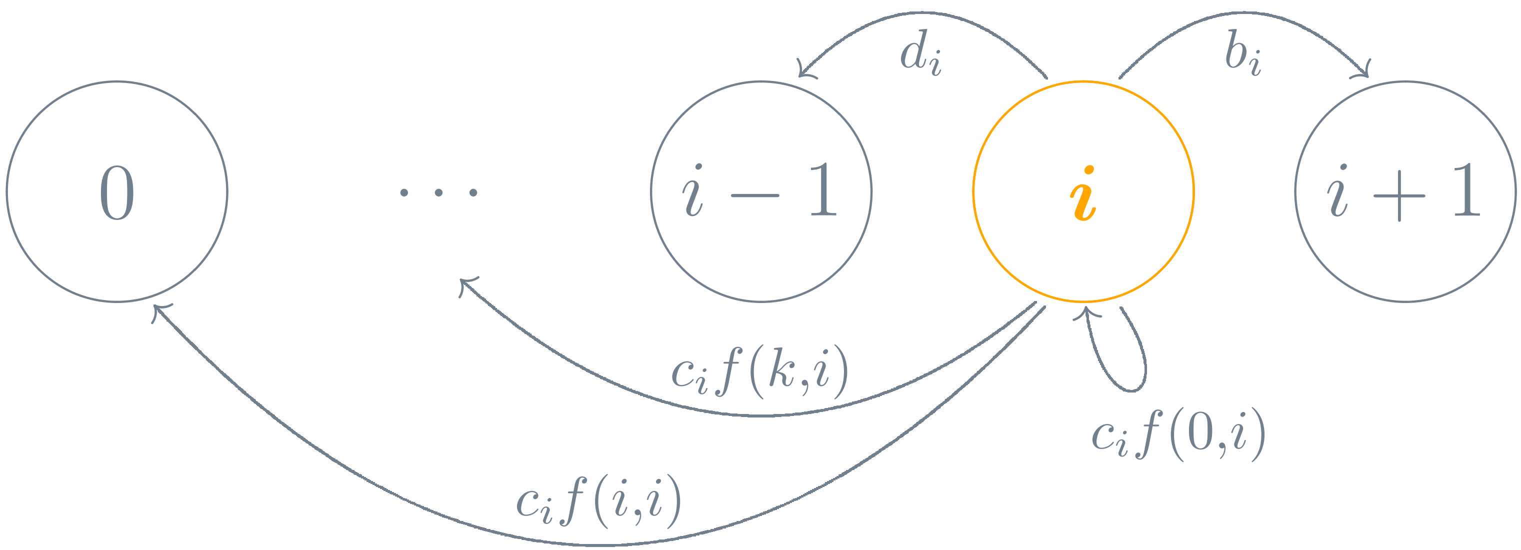

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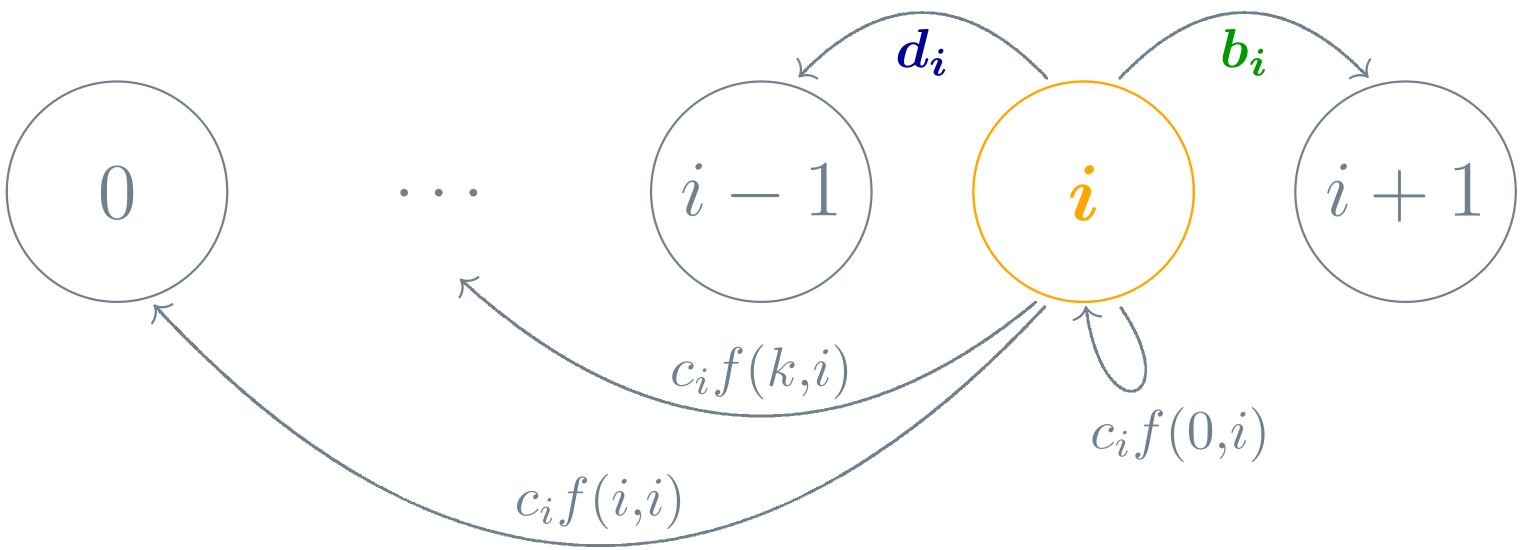

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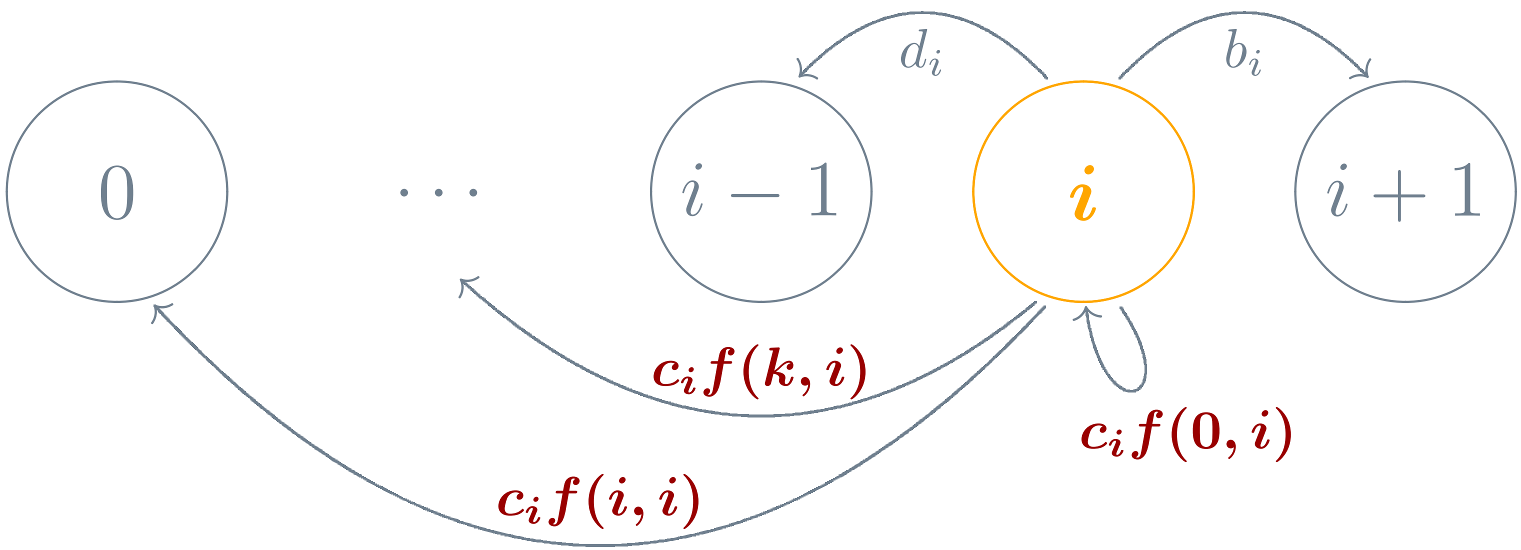

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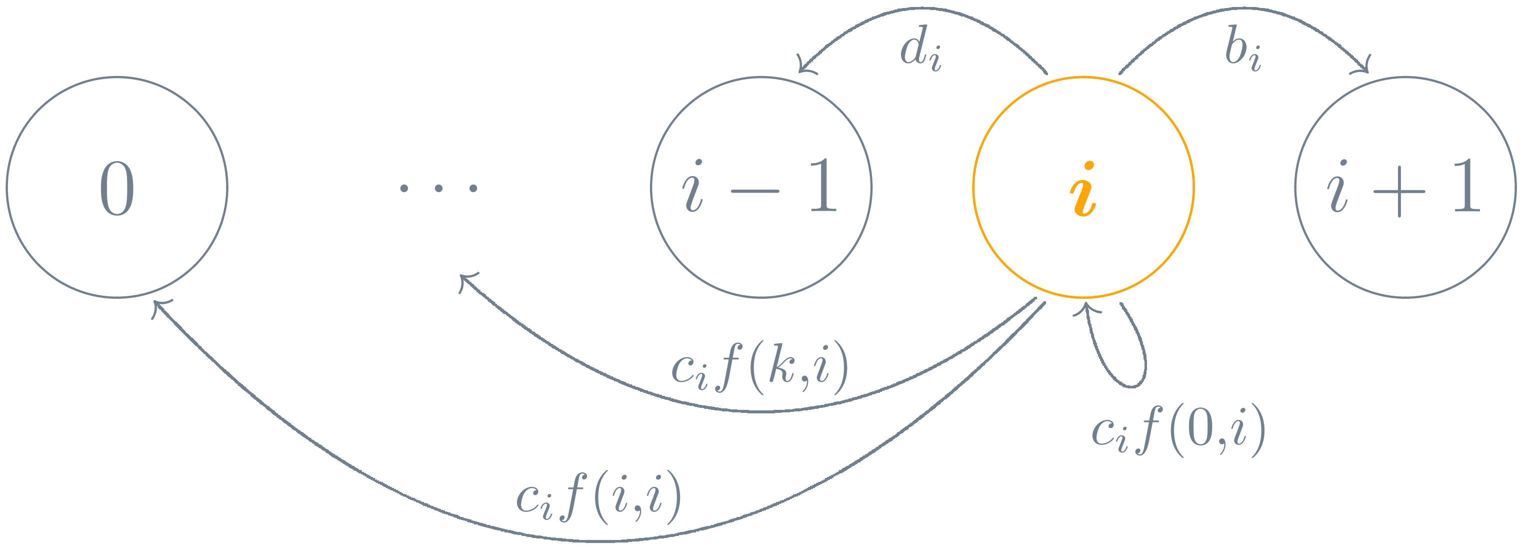

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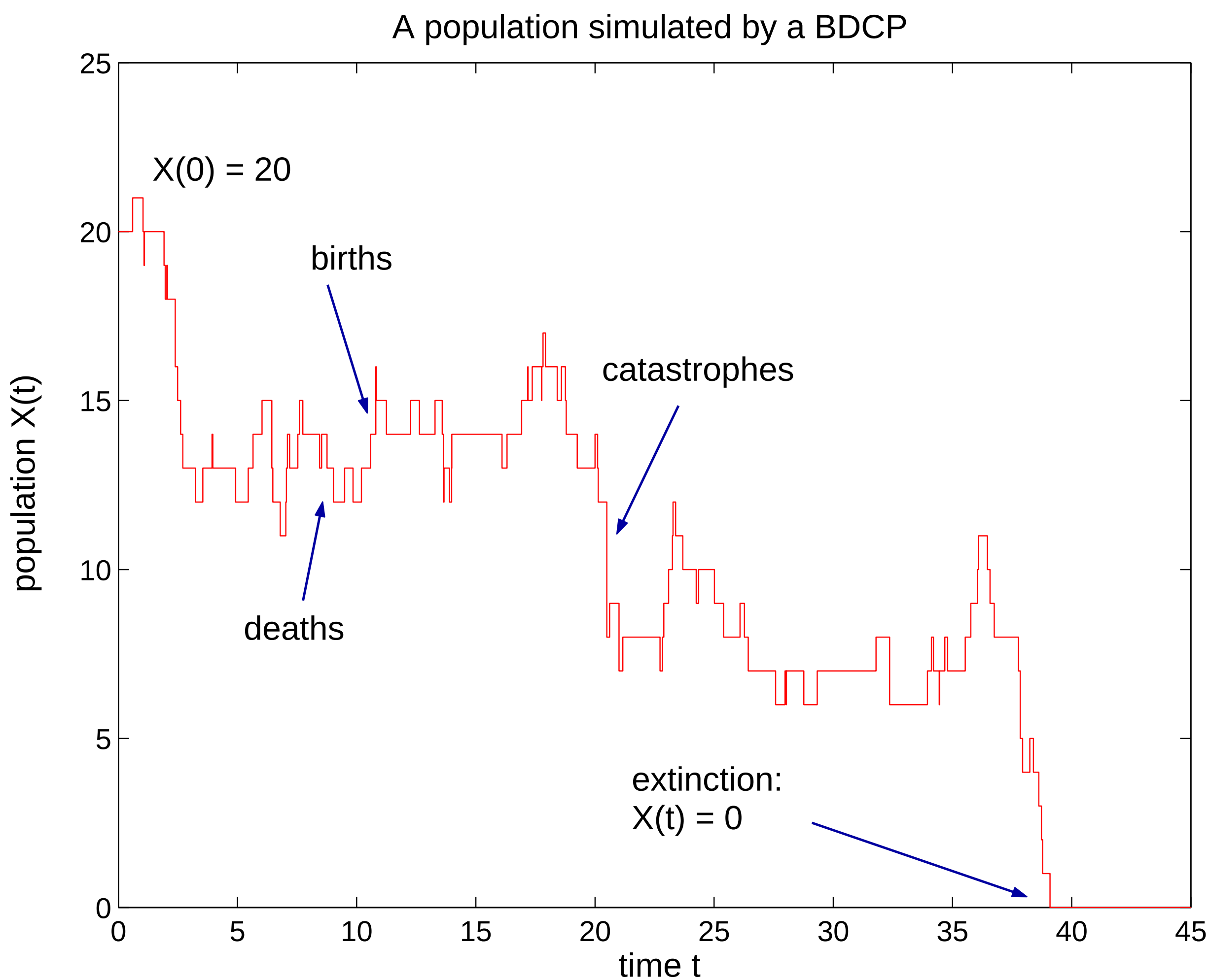
- Catastrophes occur at rate c_i, killing k with probability $f(k,i)$.
- We express these transitions as rates q_{ij} from i to j individuals. Rate $q_{ii} = -\sum_{j \neq i} q_{ij}$.

A simulation example



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For **bounded** populations, extinction is certain: Solve

$$\mathrm{M}\boldsymbol{\tau} = -1.$$

M is the matrix of transition rates q_{ij}, with i, j restricted to between 1 and the upper bound N.

Unbounded populations

Even if there is no such bound N, extinction may be certain. Then, τ_i is the minimal, non-negative solution to

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In some cases, there are analytic solutions, but often this isn't possible—we need to find good approximations.

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Our goal is to solve the problem for the truncated process, and ensure it's a good truncation.

Reflecting vs absorbing boundaries

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And those of absorbing boundaries:

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- + always underestimates τ_i;
- + can see if A is suitable.

Cons

- – apparently unrealistic;
- – less easy to get approx. τ_i.

The character of solutions

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(For τ_i, set $\theta = 0$, $\gamma_i = 1$, $z_0 = 0$.) All solutions have the form

$$z_i = a_i \kappa - b_i,$$

where $\{a_i\}$ and $\{b_i\}$ are unique sequences (see [And91] or [CP04]) and $\kappa = z_1$.

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$\{\kappa_A\}$ is increasing, so $\kappa_A \uparrow \tau_1$ as $A \to \infty$, so if κ_A is close to τ_1, approximate

$$\tau_i \approx a_i \kappa_A - b_i, \quad 0 \leq i < A.$$

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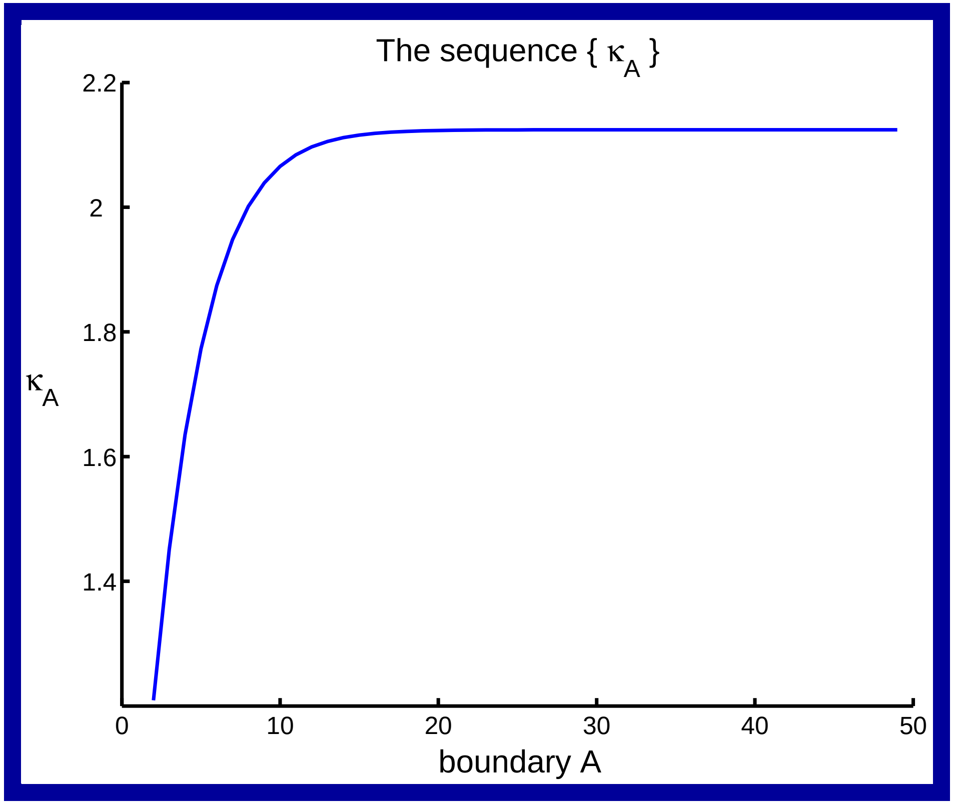
BUT, we can check for apparent convergence of κ_A to some limit as A gets large.

Convergence of κ_A



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κ_A appears
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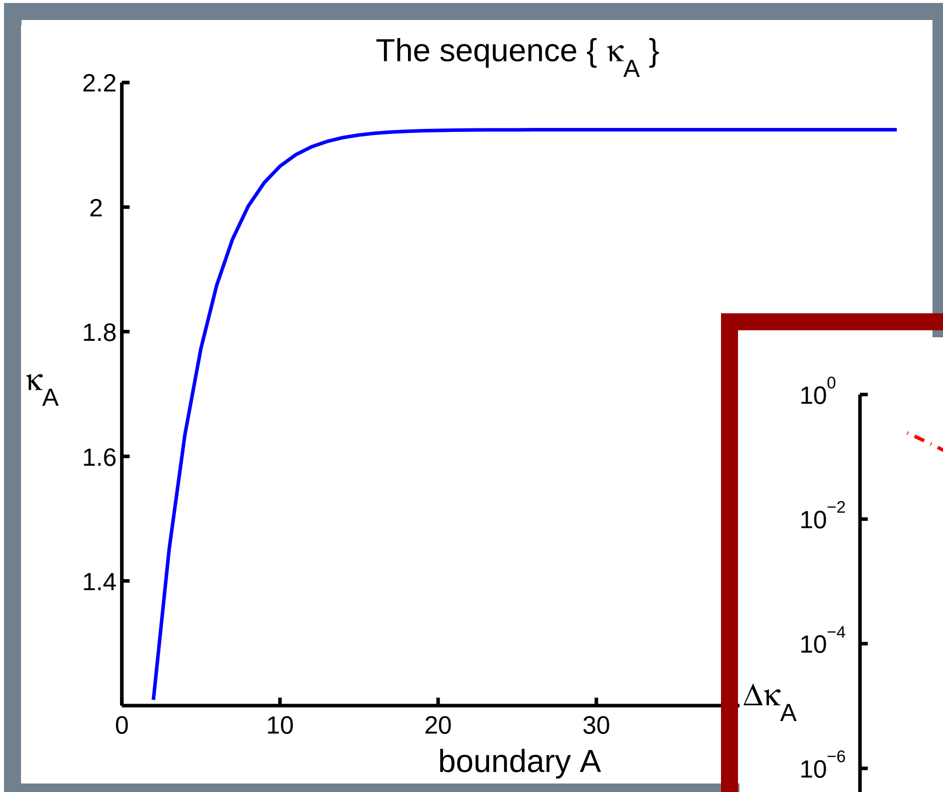

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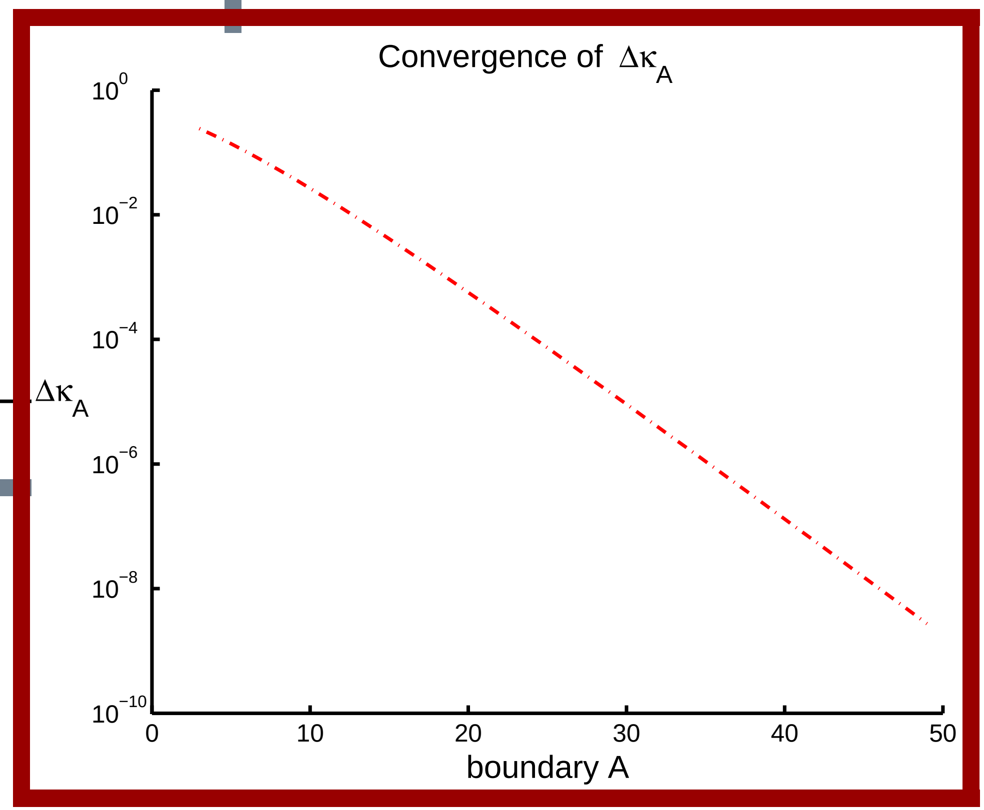

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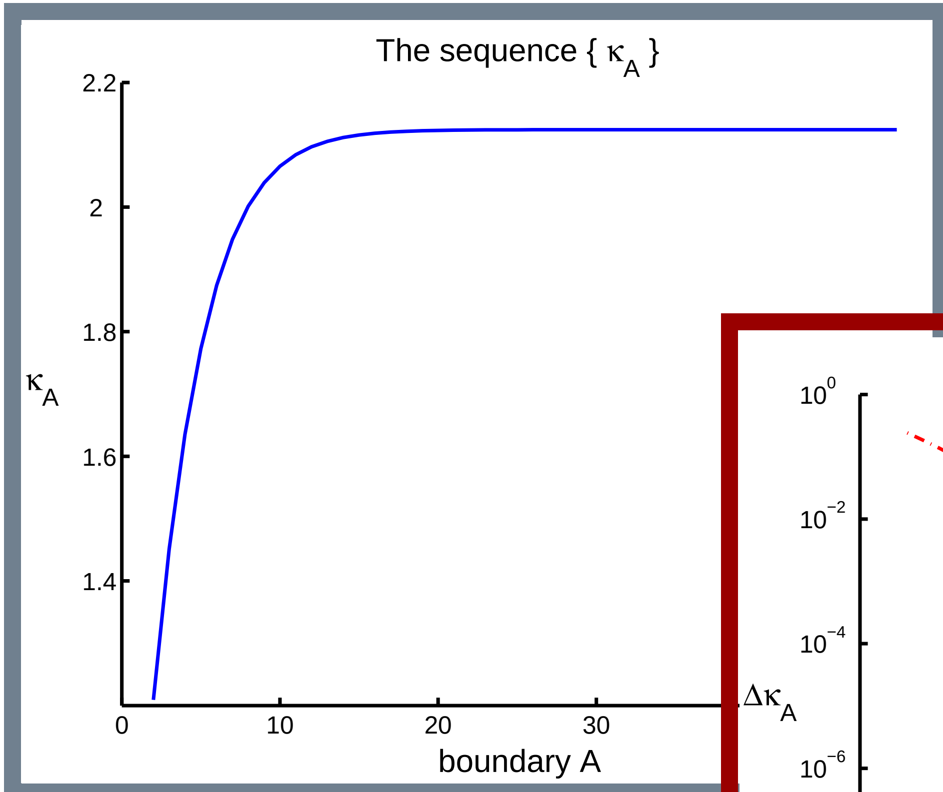

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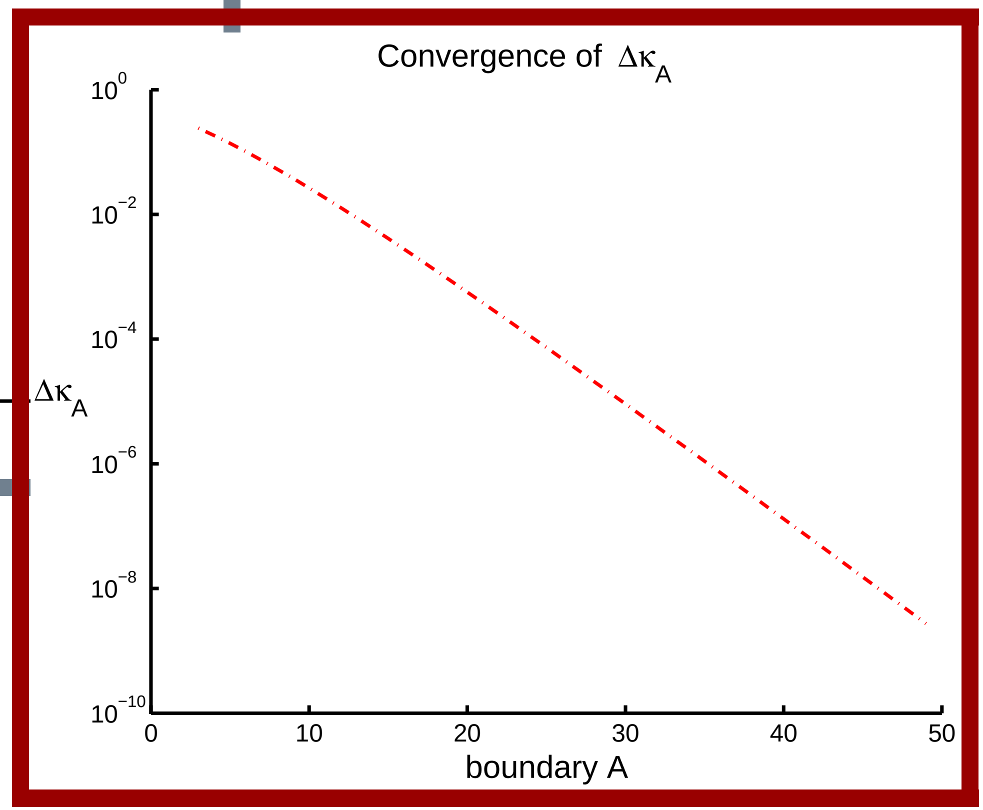

$\Delta\kappa_A = \kappa_{A+1} - \kappa_A$

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What if we take the (already) bounded process and make the boundary absorbing?

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Compare absorbing and reflecting boundaries to see if the boundary plays a significant role in the population process.

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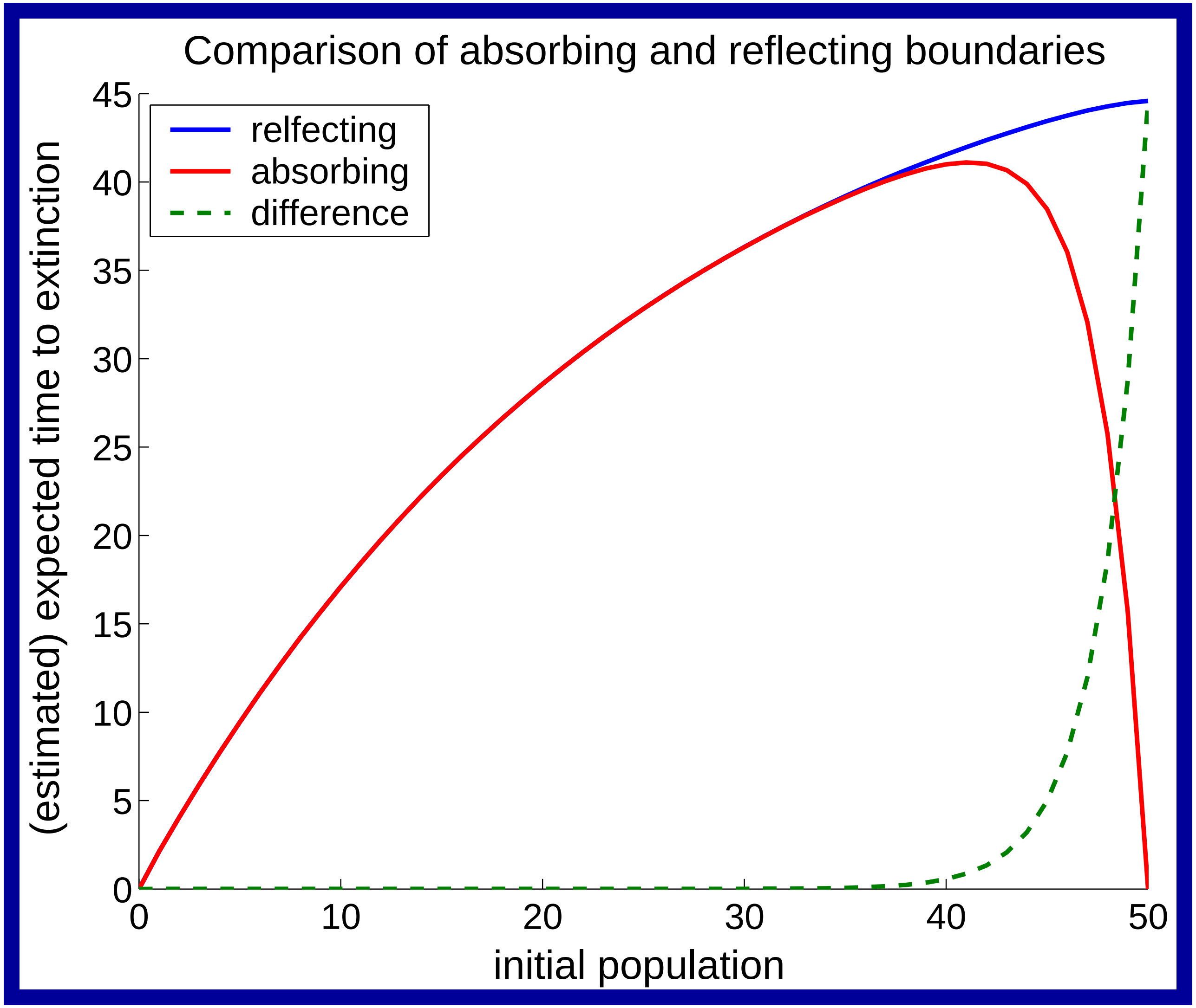

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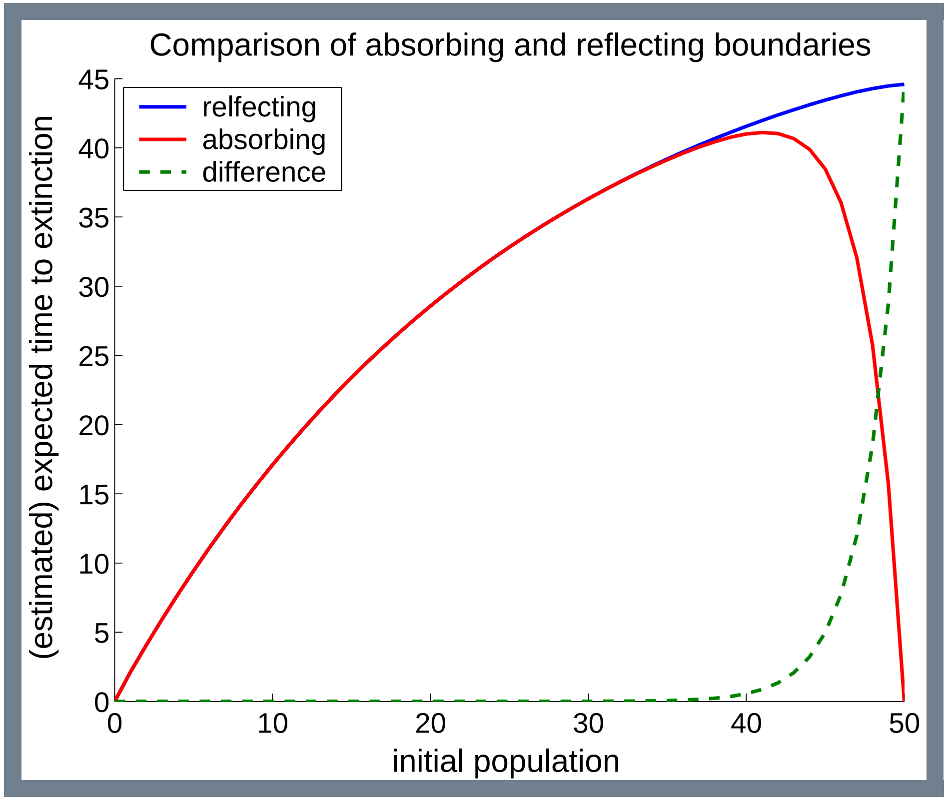

The difference between the two results is the contribution to the extinction time, in the reflecting case, from the event that the population hits N **at least once** before extinction.

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  - Reflecting boundary: simple, but may give over- *or* under-estimates.
  - Absorbing boundary: counterintuitive, but reliably underestimates persistence.
- Both can be compared to assess the effect of the boundary on persistence.

Thanks

- AMSI and ICE-EM.
- Phil Pollett and Hugh Possingham (advisors).
- AUSTRALIAN RESEARCH COUNCIL
  Centre of Excellence for Mathematics and Statistics of Complex Systems

References

[And91] W.J. Anderson. *Continuous-Time Markov Chains: An Applications-Oriented Approach*, Springer-Verlag, New York, 1991.

[CP04] B.J. Cairns and P.K. Pollett (2004). Approximating measures of persistence in a general class of population processes. (Submitted for publication.)

See also: http://www.maths.uq.edu.au/∼bjc/talks.html