Polynomial interpolation on the sphere, reproducing kernels and random matrices

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Topics

- Background and motivation
- Random Gram matrices and related distributions
- Indicative “results”
- Related conjectures

Polynomials on the unit sphere

Sphere $\mathbb{S}^2 := \{x \in \mathbb{R}^3 \mid \sum_{k=1}^{3} x_k^2 = 1\}$.

$\mathbb{P}_t(\mathbb{S}^2)$: spherical polynomials of degree at most t.

$\mathbb{H}_s^*$: homogeneous spherical harmonics of degree s, dimension $N_s = 2s + 1$.

$\mathbb{P}_t(\mathbb{S}^2) = \bigoplus_{\ell=0}^{t} \mathbb{H}_s^*$ has spherical harmonic basis

$$\{Y_{s,k} \mid s \in 0 \ldots t, k \in 1 \ldots N_s\},$$

and dimension $m_t := \sum_{s=0}^{t} N_s = (t+1)^2$.

(Reimer 1990)

Spherical polynomials as a RKHS

$\mathbb{P}_t(\mathbb{S}^2)$ is a reproducing kernel Hilbert space with inner product

$$\langle p, q \rangle := \int_{\mathbb{S}^2} p(y) q(y) d\omega(y),$$

with ω the surface measure on $\mathbb{S}^2$.
The reproducing kernel of $\mathbb{P}_t(\mathbb{S}^2)$ is

$$g_t(x, y) = \sum_{s=0}^{t} \sum_{k=1}^{N_s} Y_{s,k}(x) Y_{s,k}(y) = \frac{t+1}{4\pi} P^{(1,0)}(x \cdot y),$$

where $P^{(\alpha,\beta)}$ is the usual notation for a Jacobi polynomial.

(Reimer and Sündermann 1985; Reimer 1990; Womersley and Sloan 2001)

Spherical polynomials as a RKHS (2)

Thus for $g_x(y) := g_t(x, y)$ we have

$g_x \in \mathbb{P}_t(\mathbb{S}^2)$ and $\langle g_x, p \rangle = p(x)$

for all $p \in \mathbb{P}_t(\mathbb{S}^2)$ and all $x \in \mathbb{S}^2$.

Thus

$$\langle g_x, g_y \rangle = g_t(x, y)$$

for all $x, y \in \mathbb{S}^2$.

(Reimer and Sündermann 1985; Reimer 1990; Womersley and Sloan 2001)

Polynomial interpolation

Let $X := (x_1, \ldots, x_{m_t}) \in (\mathbb{S}^2)^{m_t}$ and define Lagrange interpolation polynomials $(\ell_1, \ldots, \ell_{m_t}) \in (\mathbb{P}_t(\mathbb{S}^2))^{m_t}$ such that $\ell_i(x_j) = \delta_{i,j}$.

X is unisolvent wrt. polynomial interpolation on $\mathbb{P}_t(\mathbb{S}^2)$, ie. X is a *fundamental system*, if and only if $\{\ell_1, \ldots, \ell_{m_t}\}$ is linearly independent.

(Reimer and Sündermann 1985; Reimer 1990; Womersley and Sloan 2001)

Gram matrix and polynomial interpolation

Given $X = (x_1, \ldots, x_{m_t})$, let $g_i := g_{x_i}$ and define the Gram matrix $G = G(X)$ such that $G_{i,j} := \langle g_i, g_j \rangle = g_t(x_i, x_j)$.

Define $\Delta = \Delta(X) := \det G(X)$.
Then it can be shown that

$$\ell_i^2(y) = \frac{\Delta(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_{m_t})}{\Delta(x_1, \ldots, x_{m_t})}.$$

(Reimer and Sündermann 1985; Reimer 1990; Womersley and Sloan 2001)

Extremal fundamental systems

Given $X = (x_1, \ldots, x_{m_t})$, define the interpolation operator $\Lambda(X) : C(\mathbb{S}^2) \to \mathbb{P}_t(\mathbb{S}^2)$ by

$$\Lambda(X) f(y) := \sum_{i=1}^{m_t} f(x_i) \ell_i(y).$$

The uniform norm of $\Lambda(X)$ is thus $\|\Lambda\|_\infty \leq \sum_{i=1}^{m_t} \|\ell_i\|_\infty$.

For the *extremal fundamental system* $\hat{X}$ which maximizes $\Delta(X)$, we therefore have $\|\Lambda\|_\infty (\hat{X}) \leq m_t$.

(Reimer and Sündermann 1985; Reimer 1990; Womersley and Sloan 2001; Sloan and Womersley 2004)

Interpolatory quadrature

Given $X = (x_1, \ldots, x_{m_t})$, define the quadrature functional $Q(X) : C(\mathbb{S}^2) \to \mathbb{R}$ by

$$Q(X)f := \int_{\mathbb{S}^2} \big(\Lambda(X)f\big)(y)d\omega(y) = \sum_{i=1}^{m_t} w_i f(x_i),$$

so

$$w_i = \int_{\mathbb{S}^2} \ell_i(y)d\omega(y),$$

We have $Gw = e$, where e is the all ones vector, and $\sum_{i=1}^{m_t} w_i = 4\pi$.

(Reimer 1994; Sloan and Womersley 2004)

Random Gram matrices

We take $\boldsymbol{X}$ to be a random $\boldsymbol{m_t}$-tuple of points independently uniformly distributed on $\mathbb{S}^{\mathbf{2}}$ and examine various distributions related to the random Gram matrix $\boldsymbol{G(X)}$:
$\boldsymbol{\Delta(X)}$, eigenvalues, quadrature weights.

Related work on random Gram matrices include: applications to combinatorics of folding and colouring (Di Francesco 1999), statistical mechanics (Hoyle and Rattray 2004), and wireless communication (Hachem, Loubaton and Najim 2005), studies of the Laguerre, Gram and Bernoulli ensembles (Roualt 2007), and studies based on kernels related to machine learning (Shawe-Taylor, Williams, Cristianini and Kandola 2002; Zhang, Kwok and Yeung 2006).

Random Gram determinant

Mean:

$$\mathbf{E}(\boldsymbol{\Delta}(\mathbf{X})) = \frac{m_t!}{(4\pi)^{m_t}}.$$

(Reimer 1997)

Upper bound:

$$\boldsymbol{\Delta}(X) \leq \left(\frac{m_t}{4\pi}\right)^{m_t}.$$

(Equality only for $t = 1$.)

(Reimer 1997; Sloan and Womersley 2004)

Random Gram eigenvalues

For given $\boldsymbol{X}$, $\textbf{trace}\, \boldsymbol{G(X)} = \boldsymbol{m_t^2/4\pi}$, so mean $\boldsymbol{\lambda = m_t/4\pi}$.
Since $\boldsymbol{G(X)}$ is positive semidefinite, $\boldsymbol{\min \lambda_{\min} = 0}$.

(Reimer and Sündermann 1985; Reimer 1997; Sloan and Womersley 2004)

Random interpolatory quadrature weights

Since $\sum_{i=1}^{m_t} w_i = 4\pi$, mean $w = 4\pi/m_t$.

(Reimer 1994; Sloan and Womersley 2004)

Also, if $w_{\min} \geq 0$ then $w_{\max} \leq 16\pi/m_t$

(Leopardi 2007 based on Reimer 2000)

Indicative "results"

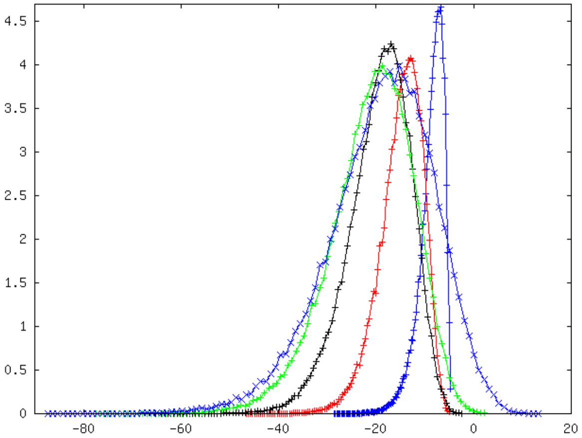
Using the Matlab code of Womersley, with Octave, including its standard pseudorandom number generator, **100 000** pseudo-Monte Carlo trials were conducted for each of degrees 1, 2, 3, 4 and 5.

Plots are histograms using 100 bins for:

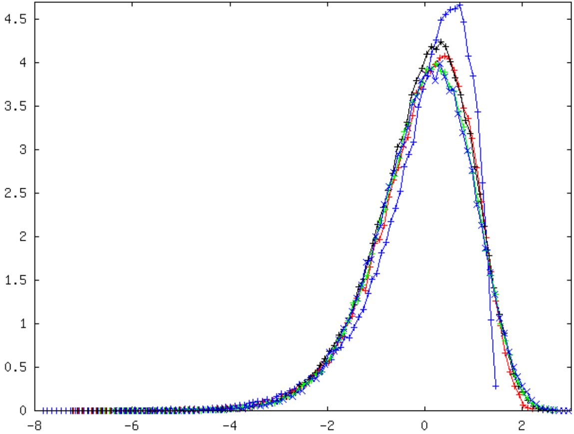
1. $\log \boldsymbol{\Delta}$;
2. $\frac{\log \boldsymbol{\Delta} - \boldsymbol{E}(\log \boldsymbol{\Delta})}{\boldsymbol{\sigma}(\log \boldsymbol{\Delta})}$.

(Sloan and Womersley 2004)

$\log \Delta$



"Normalized" $\log \Delta$



Some asymptotics

Given $\tilde{P}^{(1,0)}(z) := P^{(1,0)}(z)/P^{(1,0)}(1)$, we have

$$\lim_{t\to\infty} \tilde{P}_t^{(1,0)}\left(\cos\frac{\theta}{t}\right) = \frac{2}{\theta}J_1(\theta),$$

where J_1 is the Bessel function of order 1.

How does this asymptotic result influence the asymptotics of $\Delta(X)$ and other statistics related to $G(X)$?

(Szego 1975; Andrews, Askey and Roy 1999; Reimer 2000; Gautschi and Leopardi 2007)

Related conjectures

- For all t, for the extremal fundamental system $\hat{X}$, all weights are positive.

  (Reimer 1994; Sloan and Womersley 2004)

- For all t, there exists an equal weight interpolatory quadrature rule (ie. a *spherical* t*-design* of size $m_t = (t+1)^2$.)

  (Korevaar and Meyers 1993; Hardin and Sloane 1996; Chen and Womersley 2006; Hesse and Leopardi 2007)

- The convergence of $\tilde{P}_t^{(1,0)}$ is monotonically increasing for $\cos\theta$ greater than the largest zero of $P_1^{(1,0)}$.

  (Gautschi and Leopardi 2007)