The asymmetric exclusion process and Askey-Wilson polynomials

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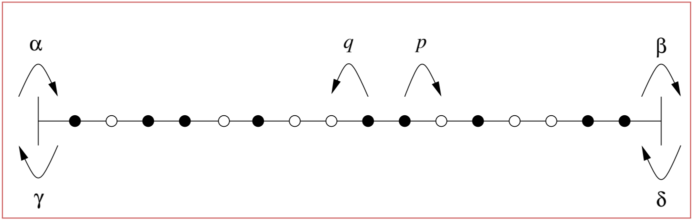
Workshop on Stochastics and Special Functions
UQ, Brisbane 2009

Partially Asymmetric Simple Exclusion Process (PASEP)



- Probability to find a configuration $\mathcal{C}$ at time t: $P(\mathcal{C}, t)$
- Time evolution given by the Master Equation:

$$\frac{\partial}{\partial t} P(\mathcal{C}, t) = \sum_{\mathcal{C}' \neq \mathcal{C}} P(\mathcal{C}', t) W(\mathcal{C}' \to \mathcal{C}) - P(\mathcal{C}, t) \sum_{\mathcal{C}' \neq \mathcal{C}} W(\mathcal{C} \to \mathcal{C}')$$

subject to some initial condition $P(\mathcal{C}, 0)$.

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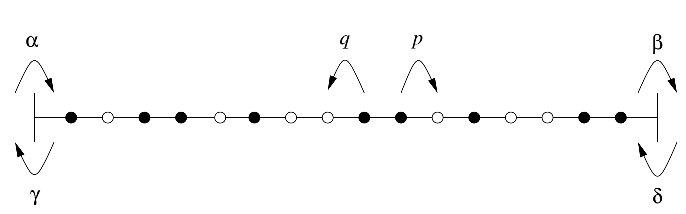

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subject to some initial condition $P(\mathcal{C}, 0)$.

Denote $\mathcal{C} = (\tau_1, \dots, \tau_N)$, $\tau_i \in \{0, 1\}$ and use unnormalised weights f:

$$P(\mathcal{C}, t) = \frac{1}{Z_N} f(\tau_1, \dots, \tau_N)$$

Theorem

$\exists$ (infinite) matrices D, E, a row vector ℓ and column vector r such that for the stationary state ($t = \infty$):

$$f(\tau_1, \dots, \tau_N) = \ell \cdot \prod_{i=1}^{N} [\tau_i D + (1 - \tau_i) E] \cdot r$$

Example:

$$f(10110100) = \ell \cdot DEDDEDEE \cdot r$$

Normalisation:

$$Z_N = \ell \cdot (D + E)^N \cdot r = \ell \cdot C^N \cdot r, \qquad C = D + E$$

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If matrices are known, it is easy to compute quantities of interest.

- Density:

$$\rho_i = \mathbb{E}[\tau_i] = \frac{1}{Z_N}\,\ell \cdot C^{i-1} D C^{N-i} \cdot r$$

- Current:

$$J_{i,i+1} = \mathbb{E}[\tau_i(1-\tau_{i+1})] = \frac{1}{Z_N}\,\ell \cdot C^{i-1} DE C^{N-i-1} \cdot r$$

In the case of TASEP ($q = 0 = \gamma = \delta$), matrices satisfy the relations:

$$DE = D + E \; (= C),$$

$$D \cdot r = \frac{1}{\beta} r,$$

$$\ell \cdot E = \frac{1}{\alpha} \ell.$$

Hence

$$J_{i,i+1} = J = \frac{1}{Z_N}\,\ell \cdot C^{i-1}(D+E)C^{N-i-1} \cdot r = \frac{Z_{N-1}}{Z_N}$$

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Matrix algebra

Master equation can be written as

$$\frac{\partial f}{\partial t} = H\, f$$

with

$$H = h_0 + \sum_{i=1}^{N-1} h_{i,i+1} + h_N$$

Example:

$$h_{i,i+1} f(\ldots 10 \ldots) = q\, f(\ldots 01 \ldots) - f(\ldots 10 \ldots).$$

or

$$h_{i,i+1}(DE) = qED - DE.$$

Example:

$$h_0(D) = \alpha E - \gamma D.$$

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Proof of Theorem.
Stationary state:

$$f(\tau_1,\ldots,\tau_N) = \ell \cdot \prod_{i=1}^{N} [\tau_i D + (1-\tau_i)E] \cdot r$$

Let D and E satisfy the relations

$$\begin{aligned}
-h_{i,i+1}(DE) &= DE - qED &= D + E \\
h_N(E)\cdot r &= (\beta D - \delta E)\cdot r &= r \\
\ell \cdot h_0(D) &= \ell \cdot (\alpha E - \gamma D) &= \ell.
\end{aligned}$$

Then $H\, f = (h_0 + \sum_{i=1}^{N-1} h_{i,i+1} + h_N)\, f$ telescopes to 0. □

Note: Different model → same matrix product formalism but with different matrix algebra.

Proof of Theorem.

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Explicit representation for TASEP

In the case $q = \gamma = \delta$:

$$D = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & \\ 0 & 0 & 1 & 1 & \\ 0 & 0 & 0 & 1 & \\ \vdots & & & & \ddots \end{pmatrix}, \qquad E = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \\ 0 & 1 & 1 & 0 & \\ 0 & 0 & 1 & 0 & \\ \vdots & & & & \ddots \end{pmatrix}$$

$$\ell = \kappa(1, a, a^2, a^3, \ldots), \qquad r = \kappa(1, b, b^2, b^3, \ldots)^T$$

where

$$a = \frac{1-\alpha}{\alpha}, \qquad b = \frac{1-\beta}{\beta}, \qquad \kappa = \frac{1}{\alpha} + \frac{1}{\beta} - \frac{1}{\alpha\beta}.$$

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$$C = D + E = \begin{pmatrix} 2 & 1 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & \\ 0 & 1 & 2 & 1 & \\ 0 & 0 & 1 & 2 & \\ \vdots & & & & \ddots \end{pmatrix}$$

Want to compute $Z_N = \ell \cdot C^N \cdot r \quad \rightarrow$ diagonalisation:

$$C \cdot e(\theta)^T = 2(1 + \cos\theta)\ e(\theta)^T$$

where

$$e(\theta)^T = \frac{1}{\sin\theta} \begin{pmatrix} \sin\theta \\ \sin 2\theta \\ \sin 3\theta \\ \vdots \end{pmatrix}$$

Elements of $e(\theta)$ are Chebyshev polynomials of the second kind.

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Due to orthonormality of Chebyshev polynomials:

$$\mathbb{I} = \frac{1}{\pi} \int_0^{2\pi} \sin^2\theta \; e(\theta)^T e(\theta) \, \mathrm{d}\,\theta$$

Hence

$$Z_N = \frac{1}{\pi} \int_0^{2\pi} \sin^2\theta \; \ell \cdot C^N \left[e(\theta)^T e(\theta) \right] \cdot r$$
$$= \frac{1}{\pi} \int_0^{2\pi} \sin^2\theta \; \left[\ell \cdot e(\theta)^T \right] \; 2^N (1+\cos\theta)^N \; \left[e(\theta) \cdot r \right] \mathrm{d}\,\theta$$

$$\ell \cdot e(\theta)^T = \frac{\kappa}{\sin\theta} \sum_{n=0}^{\infty} a^n \sin(n+1)\theta = \frac{\kappa}{(1 - a\,e^{\mathrm{i}\,\theta})(1 - a\,e^{-\mathrm{i}\,\theta})}$$

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$$Z_N = \frac{1}{\pi}\int_0^{2\pi} \sin^2\theta \, \frac{\kappa^2\, 2^N (1+\cos\theta)^N}{(1-a e^{i\theta})(1-a e^{-i\theta})(1-b e^{i\theta})(1-b e^{-i\theta})}.$$

- Similar computations for density, current and other observables
- Asymptotic analysis $\rightarrow$ non-analytic dependence on a and b
- Phase transitions

Next: allow backhopping: $q \neq 0$.

$$Z_N = \frac{1}{\pi}\int_0^{2\pi} \sin^2\theta \, \frac{\kappa^2\, 2^N (1+\cos\theta)^N}{(1-ae^{i\theta})(1-ae^{-i\theta})(1-be^{i\theta})(1-be^{-i\theta})}.$$

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Diagonalise $C = D + E \quad \rightarrow \quad$ Find $e(\theta)$ such that

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Three-term recurrence:

$$2\cos\theta\, \tilde{e}_n(\theta) = (1-q^n)(1-abq^{n-1})\tilde{e}_{n-1}(\theta) + (a+b)q^n \tilde{e}_n(\theta) + \tilde{e}_{n+1}(\theta)$$

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Three term recurrence for Al-Salam-Chihara orthogonal polynomials.

Orthogonality:

$$\frac{1}{2\pi}\frac{(q,ab;q)_\infty}{(q,ab;q)_n}\int_0^\pi \frac{(e^{2i\theta},e^{-2i\theta};q)_\infty}{(ae^{i\theta},ae^{-i\theta},be^{i\theta},be^{-i\theta};q)_\infty}\,\tilde{e}_n(\theta)\tilde{e}_m(\theta)\,d\theta=\delta_{n,m}$$

Normalisation:

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Next: allow left exit and right entrance ($\gamma \neq 0$, $\delta \neq 0$).

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Next: allow left exit and right entrance ($\gamma\neq 0$, $\delta\neq 0$).

Same procedure results in three-term recurrence for Askey-Wilson orthogonal polynomials.

$$\tilde{e}_n(\theta) = \frac{(ab, ac, ad; q)_n}{a^n} \sum_{k=0}^{\infty} \frac{(q^{-n}, q^{n-1}abcd, a\,e^{i\theta}, a\,e^{-i\theta}; q)_k}{(ab, ac, ad, q; q)_k} q^k.$$

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Decay rates

Decay $P(\mathcal{C},t) - P(\mathcal{C},\infty) \propto e^{-\Lambda t}$.

$$\Lambda(\{z_j\}) = \alpha + \beta + \gamma + \delta + \sum_{j=1}^{L-1} \frac{(q-1)^2 z_j}{(1-z_j)(qz_j-1)}$$

$$\left[\frac{qz_j - 1}{1 - z_j}\right]^{2L} K(z_j) = \prod_{l \neq j}^{L-1} \left[\frac{qz_j - z_l}{z_j - qz_l}\right] \left[\frac{q^2 z_j z_l - 1}{z_j z_l - 1}\right]$$

with $K(z) = \tilde{K}(z,\alpha,\gamma)\tilde{K}(z,\beta,\delta)$ and

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Integral equation

Taking log gives:

$$Y_L(z) := g(z) + \frac{1}{L} k(a, b, c, d; z) + \frac{1}{L} \sum_{l=1}^{L-1} \log S(z_l, z)$$

Curve: $Y_L(z_j) = -\pi + \frac{2\pi j}{L}$.

Now use Cauchy:

$$\frac{1}{L} \sum_{j=1}^{L-1} h(z_j) = \oint_{C_1+C_2} \frac{\mathrm{d}\, z}{2\pi\, \mathrm{i}} h(z) \cot\left(\frac{1}{2} L Y_L(z)\right)$$

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Given Y_L we can compute Λ_1 (and other quantities of interest). To find Y_L we expand

$$Y_L(z) = \sum_{n=0}^{\infty} L^{-n} y_n(z), \qquad \xi = z_c + \sum_{n=1}^{\infty} L^{-n}(\delta_n + \mathrm{i}\,\eta_n),$$

Full solution for $y_1(z)$:

$$\begin{aligned} y_1(z) &= -\mathrm{i}\ln\left[-\frac{z}{z_\mathrm{c}}\frac{1-z_\mathrm{c}^2}{1-z^2}\right] + \kappa_1 - \mathrm{i}\ln(ab) - \lambda_1\ln(-z_\mathrm{c}) \\ &\quad - \mathrm{i}\ln\left[\frac{(-c/z;q)_\infty(-cz;q)_\infty(-z/a;q)_\infty(-qaz_\mathrm{c};q)_\infty}{(-c/z_\mathrm{c};q)_\infty(-cz_\mathrm{c};q)_\infty(-z_\mathrm{c}/a;q)_\infty(-qaz;q)_\infty}\right] \\ &\quad - \mathrm{i}\ln\left[\frac{(-d/z;q)_\infty(-dz;q)_\infty(-z/b;q)_\infty(-qbz_\mathrm{c};q)_\infty}{(-d/z_\mathrm{c};q)_\infty(-dz_\mathrm{c};q)_\infty(-z_\mathrm{c}/b;q)_\infty(-qbz;q)_\infty}\right] \\ &\quad + \lambda_1\ln\left[\frac{(qz_\mathrm{c}/z;q)_\infty(qzz_\mathrm{c};q)_\infty^2}{(qz/z_\mathrm{c};q)_\infty(qz_\mathrm{c}^2;q)_\infty^2}\right] + \lambda_1\ln\left(\frac{z-z_\mathrm{c}^{-1}}{z_\mathrm{c}-z_\mathrm{c}^{-1}}\right), \\ z_\mathrm{c} &= -1/\sqrt{ab} \end{aligned}$$

$$\Lambda_1 = -D_+ - D_- + 2\sqrt{D_+ D_-} - \sqrt{D_+ D_-}\frac{\pi^2}{L^2} + \mathcal{O}(L^{-3}).$$

with

$$D_\pm = (1-q)\frac{\rho^\pm(1-\rho^\pm)}{\rho^+ - \rho^-}$$

given in terms of low and high density phase densities

$$\rho^- = \frac{1}{1+a}, \qquad \rho^+ = \frac{b}{1+b}$$

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- One collective mode: kink or domain wall

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