Primes, queues and random matrices

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Outline

- Counting primes
- Counting Riemann zeros
- Random matrices and their predictive powers
- Queues

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Recent Perspectives in Random Matrix Theory and Number Theory

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Organisers: Francesco Mezzadri (*University of Bristol*) Nina Snaith (*University of Bristol*)

A school run by **The European Commission Research Training Network** - **Mathematical Aspects of Quantum Chaos**

in association with the Newton Institute programme entitled **Random Matrix Approaches in Number Theory**

Draft Programme | **Participants** | **Workshop Photograph**

Lecturers

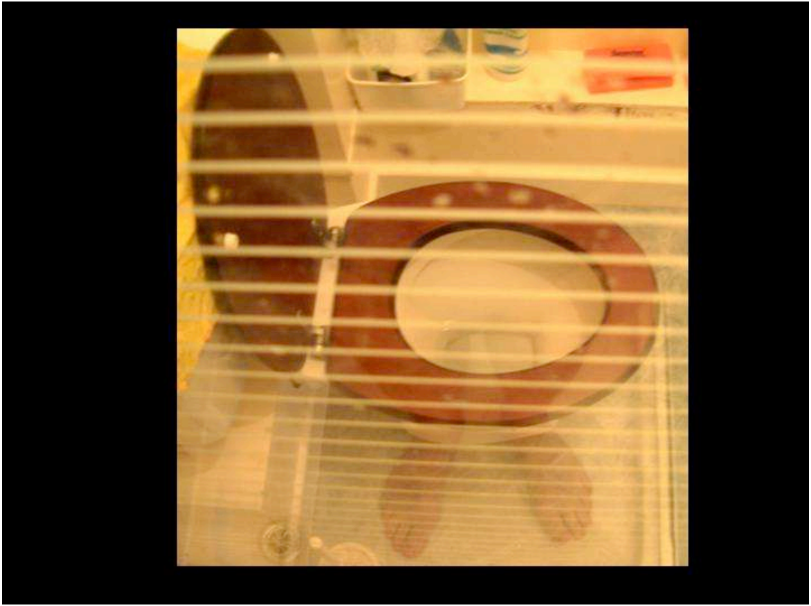
Estelle Basor, Michael Berry, Eugene Bogomolny, Oriol Bohigas, Brian Conrey, Dan Goldston, David Farmer, Peter Forrester, Yan Fyodorov, Roger Heath-Brown, Shinobu Hikami, Chris Hughes, Jonathan Keating, Philippe Michel, Michael Rubinstein

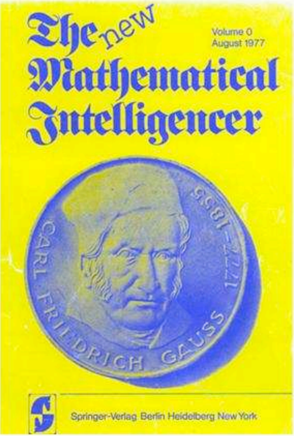
Quantum chaology and zeta

Michael Berry
University of Bristol

http://www.phy.bris.ac.uk/staff/berry_mv.html

M V Berry and J P Keating 1999
The Riemann zeros and eigenvalue asymptotics
SIAM review **41** 236-266





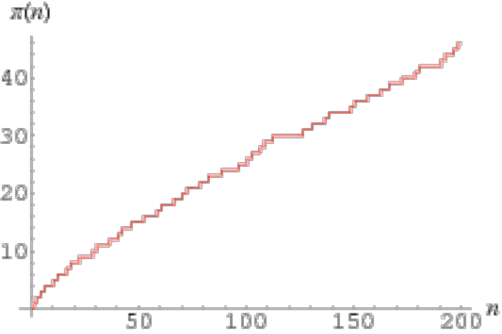
The First 50 Million Prime Numbers*

Don Zagier

I would like to tell you today about a subject which, although I have not worked in it myself, has always extraordinarily captivated me, and which has fascinated mathematicians from the earliest times until the present - namely, the question of the distribution of prime numbers.

There are two facts about the distribution of prime numbers of which I hope to convince you so overwhelmingly that they will be permanently engraved in your hearts. The first is that, despite their simple definition and role as the building blocks of the natural numbers, the prime numbers belong to the most arbitrary and ornery objects studied by mathematicians: they grow like weeds among the natural numbers, seeming to obey no other law than that of chance, and nobody can predict where the next one will sprout. The second fact is even more astonishing, for it states just the opposite: that the prime numbers exhibit stunning regularity, that there are laws governing their behaviour, and that they obey these laws with almost military precision.

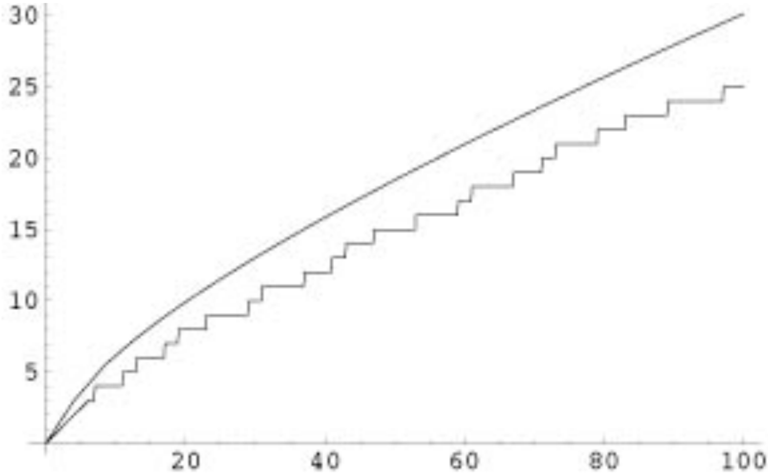
The number of primes less than a given number of primes



X	primes $< X$	%
10^2	25	25%
10^3	168	16.8%
10^4	1,229	12.3%
10^5	9,592	9.6%
10^6	78,498	7.8%

A good guess (Gauss)

$$\Pi(X) := (\#\ \text{primes} \; < X) \sim \int_0^X \frac{dt}{\log t} =: \mathrm{Li}(x) \sim \frac{X}{\log X}$$



Gauss's function compared to the true number of primes

Aside: Sign changes of $\mathrm{Li}(X) - \Pi(X)$

- Graphically $\mathrm{Li}(X) - \Pi(X) > 0$
- In 1914 Littlewood proved $\mathrm{Li}(X) - \Pi(X)$ changes sign infinitely often.
- Skewes in 1933 proved that this must happen for $X < 10^{10^{10^{10^{34}}}}$, which is known as Skewes' number
- Present day results gives $X < 1.39 \times 10^{316}$.
- Proved by Sarnak and Rubinstein that proportion of integers for which $\mathrm{Li}(X) - \Pi(X) < 0$ is $\approx 2.6 \times 10^{-7}$.

The (Riemann) zeta function

Euler (1707–1783) defined the zeta function

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots$$

and showed that

$$\zeta(s) = \prod_{\text{primes } p} \frac{1}{(1 - p^{-s})}$$

- Riemann (1826–1866) used $\zeta(s)$ for complex s to study $\Pi(X)$.
- In 1859 he published the paper ‘On the number of prime numbers less than a given quantity’.

Main points from Riemann’s paper (after Edwards)

- $\zeta(s)$ was analytically continued from $\mathrm{Re}(s) > 1$ to all $s \neq 1$, and a functional equation relating $\zeta(s)$ to $\zeta(1-s)$ obtained.

- Shows $\zeta(s)$ has 'trivial' zeros of $\zeta(s)$ for $s = -2, -4, -6, \ldots,$ other zeros (Riemann zeros) confined to $0 \leq \mathrm{Re}(s) \leq 1$.
- The entire function $\xi(t) = \Gamma(s)\frac{1}{2}s(s-1)\pi^{-s/2}\zeta(s)$, $s = \frac{1}{2} - it$ was introduced. Functional equation for $\zeta(s)$ equivalent to $\xi(t) = \xi(-t)$ and $\xi(t)$ real for t real.
- $\xi(t)$ vanishes at Riemann zeros of $\zeta(\frac{1}{2} - it)$ only.

- Motivated by the fact that

$$\log \zeta(s) = \sum_{\text{primes } p} \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{p^{ns}}$$

the functional equation

$$\begin{aligned} J(x) &= \#\text{ primes } < x \\ &\quad + \frac{1}{2} \#\text{ primes squared } < x \\ &\quad + \frac{1}{3} \#\text{ primes cubed } < x + \cdots \end{aligned}$$

was considered, and the formula

$$J(x) = \mathrm{Li}(x) - \sum_{\substack{\alpha : \xi(\alpha)=0 \\ \mathrm{Re}(\alpha)>0}} \left(\mathrm{Li}(x^{1/2+i\alpha}) + \mathrm{Li}(x^{1/2-i\alpha}) \right)$$

$$+ \int_x^{\infty} \frac{dt}{t(t^2-1)\log t} + \log \xi(0)$$

obtained.

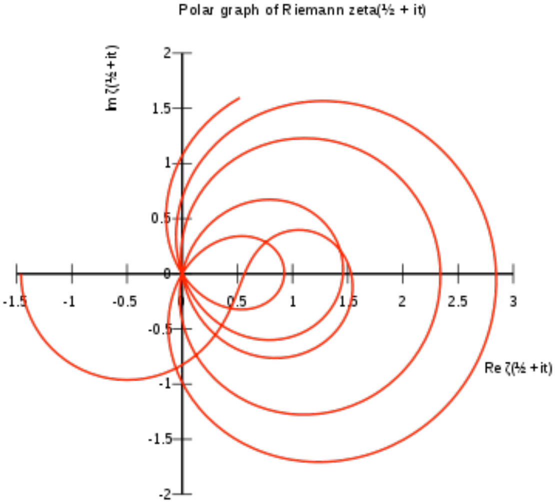
- Möbius inversion of this formula is used to show

$$\Pi(x) = \sum_{n=1}^{N} \frac{\mu(n)}{n} \mathrm{Li}(x^{1/n}) + \sum_{n=1}^{N} \sum_{\alpha:\xi(\alpha)=0} \mathrm{Li}(x^{(1/2+i\alpha)/n})$$

  Here N is such that $x^{1/(N+1)} < 2$. The Möbius function $\mu(n)$, $(= 0, \pm 1)$, appears in

$$\frac{1}{\zeta(s)} = \sum_{n} \frac{\mu(n)}{n^s}.$$

- Riemann remarks that the number of roots of $\xi(t) = 0$ whose real parts lie between 0 and T is approximately $(T/2\pi)(\log(T/2\pi) - 1)$ and says **it is very probable that all the roots are real**. This is the Riemann hypothesis.


Polar graph of Riemann zeta(½ + it)
Im ζ(½ + it)
Re ζ(½ + it)
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5
3

Implication of the Riemann hypothesis

- The statement

  $$(\#\mathrm{primes}\ < x) \sim \mathrm{Li}(x) \quad (\sim \frac{x}{\log x})$$

  which is referred to as the Prime number theorem is equivalent to the statement that no zeros of $\zeta(s)$ lie on the boundary $\mathrm{Re}(s) = 0$, $\mathrm{Re}(s) = 1$ of the critical strip $0 \leq \mathrm{Re}(s) \leq 1$. Proved by Hadamard and de la Vallee Poussin in 1896.
- van Koch (1901) showed that the Riemann hypothesis is equivalent to

  $$(\#\mathrm{primes}\ < x) = \mathrm{Li}(x) + \mathrm{O}(x^{1/2} \log x).$$

Dual problem: Counting the Riemann zeros

- Riemann gave the formula

$$N(E) = \underbrace{1 - \frac{1}{2\pi} E \log \pi - \frac{1}{\pi} \mathrm{Im} \log \Gamma\left(\frac{1}{4} - \frac{iE}{2}\right)}_{\bar{N}(E)} \underbrace{- \frac{1}{\pi} \mathrm{Im} \log \zeta\left(\frac{1}{2} - iE\right)}_{N_{\mathrm{osc}}(E)}$$

- Substituting for $\log(1/2 - iE)$ using the Euler product (illegal) gives

$$N_{\mathrm{osc}}(E) = -\frac{1}{\pi} \sum_{\mathrm{primes}\ p} \sum_{m=1}^{\infty} \frac{\exp(-(m/2)\log p)}{m} \sin(Em \log p)$$

- Compare with Gutzwiller trace formula for oscillating part of a quantum spectrum from classical data:

$$N_{\mathrm{osc}}(E) \underset{\hbar \to 0}{\sim} \frac{1}{\pi} \sum_{\tilde{p}} \sum_{m=1}^{\infty} \frac{\exp(-(m/2)\lambda_{\tilde{p}} T_{\tilde{p}})}{k} \sin\left(\frac{m}{\hbar} S_{\tilde{p}}(E) - \frac{\pi m}{2} \mu_{\tilde{p}}\right)$$

dictionary

	quantum	Riemann
oscillations	primitive orbits p, repetitions m	primes p, integers m
dimensionless actions	$\frac{mS_p}{\hbar}$	$mt \log p$
periods	mT_p	$m \log p$
stabilities	$\frac{1}{2}\lambda_p T_p$	$\frac{1}{2}\log p \Rightarrow \lambda_p = 1$
asymptotics	$\hbar \to 0$	$t \to \infty$

Here $t = E$. The fact that the prefactor is $1/\pi$ not $2/\pi$ indicates that the primitive orbits are unique and in particular there is no time reversal symmetry. The instability exponent $\lambda_{\tilde{p}} = 1$ indicates a uniformly chaotic system. Note an overall minus sign discrepancy.

Predictive powers

- For $E \to \infty$ the spectrum of a chaotic quantum system with no time reversal symmetry has the same statistical properties as the eigenvalues of a large complex Hermitian or complex unitary matrix.
- Define $\Lambda(z) = \prod_{l=1}^{N}(e^{i\theta_l} - z)$ — characteristic polynomial for matrices in $U(N)$
- Keating and Snaith hypothesized

$$\begin{array}{ccc} \text{value distribution} & & \text{value distribution} \\ \log \zeta(1/2 + it) & \underset{\log(t/2\pi)=N}{\sim} & \log \Lambda(z) \end{array}$$

Both sides can be computed.

- LHS A result of Selberg gives

$$\lim_{T\to\infty}\frac{1}{T}\left|\left\{t : T \le t \le 2T,\ \ \frac{\log\zeta(1/2+it)}{\sqrt{(1/2)\log\log T}} \in B\right\}\right|$$
$$= \frac{1}{2}\iint_B e^{-(x^2+y^2)/2}dxdy$$

- RHS

$$P(s,t) = \left\langle \delta\Big(s - \mathrm{Re}\ \underbrace{\log\Lambda(-1)}_{\sum_{j=1}^N \log|1+e^{i\theta_j}|}\ \Big)\delta\Big(t - \mathrm{Im}\ \underbrace{\log\Lambda(-1)}_{(1/2)\sum_{j=1}^N \theta_j}\ \Big)\right\rangle_{U(N)}$$

$\Rightarrow$

$$\hat{P}(k,l) = \int_{-\infty}^{\infty} ds\, e^{iks}\int_{-\infty}^{\infty} dt\, e^{ilt} P(s,t)$$
$$= \left\langle \prod_{j=1}^N e^{il\theta_j/2}|1+2\cos\theta_j|^{ik/2}\right\rangle_{U(N)} \sim e^{-\log N(k^2+l^2)/4}$$

- Keating and Snaith also hypothesized:

$$\begin{matrix}\text{value distribution} \\ |\zeta(1/2+it)|\end{matrix} \underset{\log(t/2\pi)=N}{\sim} \begin{matrix}\text{value distribution} \\ |\Lambda(z)|\end{matrix} \times \begin{matrix}\text{num.theoretic} \\ \text{factor}\end{matrix}$$

- Implies

$$\frac{1}{(\log T)^{a^2}} \frac{1}{T} \int_0^T |\zeta(1/2+it)|^{2a}\, dt \sim f(a) \times \begin{matrix}\text{num.theoretic} \\ \text{factor}\end{matrix}$$

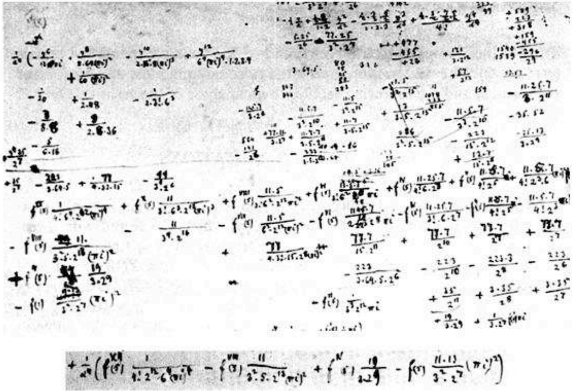
with

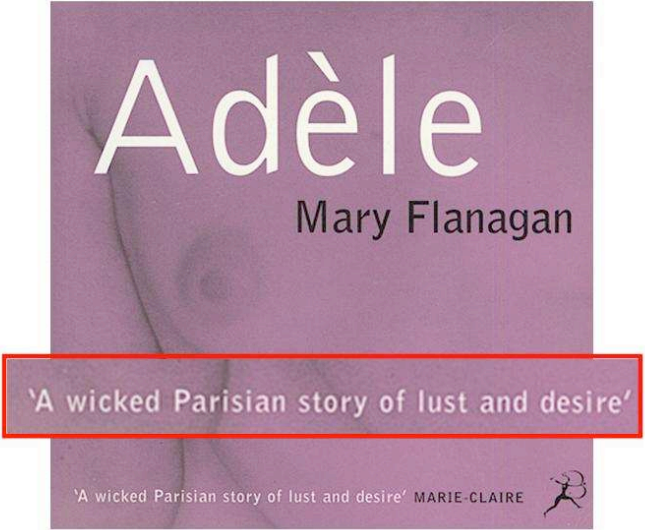
$$f(a) = \frac{G^2(a+1)}{G(2a+1)}$$

opposite intellectual current :
Riemann $\longrightarrow$ quantum

from the ***Riemann-Siegel*** formula

accurate method for calculating $\zeta(s)$ on the critical line




Adèle
Mary Flanagan
'A wicked Parisian story of lust and desire'
'A wicked Parisian story of lust and desire' MARIE-CLAIRE

Queues and random matrices

- Matrix of service times $[x_{jk}]_{\substack{j=1,\dots,p \\ k=1,\dots,n}}$ for p jobs and n queues in series
- T_{jk} — time it takes job j to exit queue k. Have

$$T_{i,j} = \max\left(T_{i,j-1}, T_{i-1,j}\right) + x_{j,k}$$

- Suppose each x_{jk} is i.i.d. Suppose the number of queues goes to infinity. Define scaled exit time of job j

$$D_j = \lim_{n\to\infty} \frac{T_{jn} - n\langle x_{11}\rangle}{\sqrt{(\langle x_{11}^2\rangle - \langle x_{11}\rangle^2)n}}$$

- Glynn-Whitt With $X = \{0 = t_0 < t_1 < \cdots < t_{k-1} < t_k = 1\}$, $\{B_i\}$ independent Brownian motions

$$D_j = \sup_X \sum_{i=0}^{j-1} (B_i(t_{i+1}) - B_i(t_i)).$$

- Baryshnikov Let $Y = [y_{l,l'}]$ be a Gaussian Hermitian complex matrix, $Y = (Z + Z^\dagger)/2$, Z i.i.d. complex standard Gaussians. Law of $\{D_j\}$ is the same as the law of the distribution of the largest eigenvalue in $Y_1, Y_2, \ldots$, where Y_j is the $j \times j$ leading minor of Y.