ON BIRTH-DEATH PROCESSES AND EXTREME ZEROS OF ORTHOGONAL POLYNOMIALS

Erik A. van Doorn
Department of Applied Mathematics
University of Twente
Enschede, The Netherlands

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outline

1. orthogonal polynomials

- definitions, notation
- zeros, orthogonalizing measure
- OP's on $[0, \infty)$

2. birth-death processes (with killing)

- definitions, notation
- decay rate
- recent results

3. extreme zeros of OP's

orthogonal polynomials

definition: $\{P_n(x),\ n = 0, 1, \ldots\}$ (monic, $\deg(P_n) = n$) is *orthogonal polynomial sequence (OPS)* if there exists (Borel) measure ψ (of total mass 1) such that

$$\int_{-\infty}^{\infty} P_n(x) P_m(x) \psi(dx) = k_n \delta_{nm}$$

with $k_n > 0$ (ψ is not necessarily unique)

Favard's theorem:

$\{P_n(x),\ n = 0, 1, \ldots\}$ is OPS $\iff$ there exist $c_n \in \mathbb{R},\ \lambda_n > 0$

such that

$$\begin{aligned} &P_n(x) = (x - c_n) P_{n-1}(x) - \lambda_n P_{n-2}(x) \\ &P_0(x) = 1, \quad P_1(x) = x - c_1 \end{aligned}$$

orthogonal polynomials

point of departure: $\{P_n(x),\ n = 0, 1, \ldots\}$ satisfies

$$P_n(x) = (x - c_n)P_{n-1}(x) - \lambda_n P_{n-2}(x)$$
$$P_0(x) = 1, \quad P_1(x) = x - c_1$$

with $c_n \in \mathbb{R},\ \lambda_n > 0$

general problem: find information on orthogonalizing measure from coefficients in recurrence relation (cf. Chihara's book)

fact: support of orthogonalizing measure is related to zeros of polynomials

approach: find information on zeros of $P_n(x)$ from coefficients in recurrence relation

zeros of orthogonal polynomials

$\{P_n(x),\ n = 0, 1, \ldots\}$ satisfies

$$P_n(x) = (x - c_n)P_{n-1}(x) - \lambda_n P_{n-2}(x)$$
$$P_0(x) = 1, \quad P_1(x) = x - c_1$$

if $a_j > 0$ and

$$T_n := \begin{pmatrix} c_1 & \lambda_2/a_2 & 0 & \cdots & \cdots \\ a_2 & c_2 & \lambda_3/a_3 & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \cdots & \cdots & \cdots & c_{n-1} & \lambda_n/a_n \\ \cdots & \cdots & 0 & a_n & c_n \end{pmatrix}$$

then

$$\det(xI_n - T_n) = P_n(x)$$

observation: zeros of $P_n(x)$ are eigenvalues of T_n

zeros of orthogonal polynomials

$\{P_n(x),\ n = 0, 1, \ldots\}$ is OPS with zeros x_{ni}

zeros of $P_n(x)$ real and distinct:

$$x_{n1} < x_{n2} < \ldots < x_{nn}$$

interlacing property:

$$x_{n+1,i} < x_{ni} < x_{n+1,i+1}$$

hence

$$\xi_i := \lim_{n\to\infty} x_{ni} \quad \text{and} \quad \sigma := \lim_{i\to\infty} \xi_i$$

exist, and

$$-\infty \le \xi_i \le \xi_{i+1} \le \sigma \le \infty$$

zeros of orthogonal polynomials

$\{P_n(x),\ n = 0, 1, \ldots\}$ is OPS with zeros x_{ni}

let

$$\xi_i := \lim_{n\to\infty} x_{ni} \quad \text{and} \quad \sigma := \lim_{i\to\infty} \xi_i$$

then

$$-\infty \le \xi_i \le \xi_{i+1} \le \sigma \le \infty$$

moreover

$$\xi_i = \xi_{i+1} \ \Rightarrow \ \xi_i = \sigma$$

similarly

$$\eta_i := \lim_{n\to\infty} x_{n,n-i+1} \quad \text{etc}$$

zeros of orthogonal polynomials

$\{P_n(x),\ n = 0, 1, \ldots\}$ is OPS with zeros x_{ni} and $\xi_i := \lim_{n\to\infty} x_{ni}$

then

$$-\infty \leq \xi_i \leq \xi_{i+1} \leq \sigma = \lim_{i\to\infty} \xi_i \leq \infty$$

and

$$\xi_i = \xi_{i+1} \ \Rightarrow \ \xi_i = \sigma$$

if $\xi_1 > -\infty$ there are three possibilities:

1. $\xi_1 < \cdots < \xi_i < \xi_{i+1} < \cdots < \sigma = \infty$
2. $\xi_1 < \cdots < \xi_i < \xi_{i+1} < \cdots < \sigma < \infty$
3. $\xi_1 < \cdots < \xi_i = \xi_{i+k}$ for some i and all $k > 0$

orthogonalizing measure

$\{P_n(x),\ n = 0, 1, \ldots\}$ is OPS satisfying

$$P_n(x) = (x - c_n)P_{n-1}(x) - \lambda_n P_{n-2}(x)$$
$$P_0(x) = 1, \quad P_1(x) = x - c_1$$

theorem: if $\xi_1 > -\infty$ there exists an orthogonalizing measure ψ for $\{P_n(x)\}$, that is,

$$\int_{-\infty}^{\infty} P_m(x)P_n(x)\psi(dx) = k_n\delta_{mn}$$

such that

$$\sigma = \infty \ \Rightarrow \ \mathrm{supp}(\psi) = \{\xi_1, \xi_2, \ldots\}$$

$$\sigma < \infty \ \Rightarrow \ \mathrm{supp}(\psi) \cap (-\infty, \sigma] = \overline{\{\xi_1, \xi_2, \ldots\}}$$

remark: ψ not necessarily unique (*Hamburger moment problem*)

orthogonalizing measure

$\{P_n(x),\ n = 0, 1, \ldots\}$ is OPS satisfying

$$P_n(x) = (x - c_n)P_{n-1}(x) - \lambda_n P_{n-2}(x)$$
$$P_0(x) = 1, \quad P_1(x) = x - c_1$$

general problem: find information on orthogonalizing measure from coefficients in recurrence relation

specific problem: find information on ξ_1 (and x_{n1}, and ξ_2) in terms of coefficients in recurrence relation

observe: $\{\tilde{P}_n(x) := (-1)^n P_n(-x),\ n = 0, 1, \ldots\}$ is OPS with $\tilde{c}_n := -c_n$ and $\tilde{\lambda}_n := \lambda_n$, hence

$$x_{nn} = -\tilde{x}_{n1}$$

zeros of orthogonal polynomials

$\{P_n(x),\ n = 0, 1, \ldots\}$ is OPS satisfying

$$P_n(x) = (x - c_n)P_{n-1}(x) - \lambda_n P_{n-2}(x)$$
$$P_0(x) = 1, \quad P_1(x) = x - c_1$$

with $c_n \in \mathbb{R},\ \lambda_n > 0$

recall: zeros of $P_n(x)$ are eigenvalues of

$$T_n = \begin{pmatrix} c_1 & \lambda_2/a_2 & 0 & \cdots & \cdots \\ a_2 & c_2 & \lambda_3/a_3 & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \cdots & \cdots & \cdots & c_{n-1} & \lambda_n/a_n \\ \cdots & \cdots & 0 & a_n & c_n \end{pmatrix}$$

where $a_j > 0$

zeros of orthogonal polynomials

theorem (Gilewicz & Leopold (1985), vD (1984,1987)):

$$x_{n1} = \max_{\boldsymbol{a}>\mathbf{0}} \min_{1\le i\le n} \left\{ c_i - a_{i+1} - \frac{\lambda_i}{a_i} \right\}$$

$$\xi_1 = \max_{\boldsymbol{a}>\mathbf{0}} \inf_{i\ge 1} \left\{ c_i - a_{i+1} - \frac{\lambda_i}{a_i} \right\}$$

where $\lambda_1 = 0$, $\boldsymbol{a} = (a_1, a_2, \ldots)$

proof: *Geršgorin discs*

zeros of orthogonal polynomials

theorem (vD (1987), Ismail & Li (1992):

$$x_{n1} = \max_{\boldsymbol{h}} \min_{1 \le i < n} \frac{1}{2} \left\{ c_i + c_{i+1} - \sqrt{(c_{i+1} - c_i)^2 + \frac{4\lambda_{i+1}}{(1-h_i)h_{i+1}}} \right\}$$

where $\boldsymbol{h} = (h_1, \ldots, h_n)$, $h_1 = 0$, $h_n = 1$, $0 < h_i < 1$ $(1 < i < n)$

$$\xi_1 = \max_{\boldsymbol{b}} \inf_{i \ge 1} \frac{1}{2} \left\{ c_i + c_{i+1} - \sqrt{(c_{i+1} - c_i)^2 + 4\lambda_{i+1}/b_i} \right\}$$

where $\boldsymbol{b} = (b_1, b_2, \ldots)$ is a *chain sequence*

proof: *ovals of Cassini*

zeros of orthogonal polynomials

theorem (vD (1987), Levenshtein (1995)):

$$x_{n1} = \min_{\boldsymbol{h} \geq \boldsymbol{0}} \left\{ \sum_{i=1}^{n} \left(h_i^2 c_i - 2 h_{i-1} h_i \sqrt{\lambda_{i+1}} \right) \right\}$$

where $\boldsymbol{h} = (h_0, \ldots, h_n)$, $h_0 = 0$, $\sum_{i=1}^{n} h_i^2 = 1$

$$\xi_1 = \inf_{\boldsymbol{h} \geq \boldsymbol{0}} \left\{ \lim_{n\to\infty} \inf \left\{ \sum_{i=1}^{n} \left(h_i^2 c_i - 2 h_{i-1} h_i \sqrt{\lambda_{i+1}} \right) \right\} \right\}$$

where $\boldsymbol{h} = (h_0, h_1, \ldots)$, $h_0 = 0$, $\sum_{i=1}^{\infty} h_i^2 = 1$

proof: *Courant-Fischer theorem/Raleigh quotients/field of values* (symmetrize T_n by suitable similarity transformation)

orthogonal polynomials on $[0, \infty)$

theorem: the following are equivalent:

(i) $\xi_1 \geq 0$

(ii) there exist numbers $\alpha_n > 0$ and $\beta_{n+1} > 0$ such that $c_1 = \alpha_1$, and, for $n > 1$,

$$c_n = \alpha_n + \beta_n$$
$$\lambda_n = \alpha_{n-1}\beta_n$$

(iii) there exist numbers $\alpha_n > 0$, $\beta_{n+1} > 0$ and $\gamma_n \geq 0$ such that $c_1 = \alpha_1 + \gamma_1$, and, for $n > 1$,

$$c_n = \alpha_n + \beta_n + \gamma_n$$
$$\lambda_n = \alpha_{n-1}\beta_n$$

orthogonal polynomials on $[0,\infty)$

$\xi_1 \geq 0 \iff$ there exist numbers $\alpha_n > 0$ and $\beta_{n+1} > 0$ such that $c_1 = \alpha_1$ and, for $n > 1$,

$$c_n = \alpha_n + \beta_n, \quad \lambda_n = \alpha_{n-1}\beta_n$$

$\iff$ there exist numbers $\alpha_n > 0$, $\beta_{n+1} > 0$ and $\gamma_n \geq 0$ such that $c_1 = \alpha_1 + \gamma_1$ and, for $n > 1$,

$$c_n = \alpha_n + \beta_n + \gamma_n, \quad \lambda_n = \alpha_{n-1}\beta_n$$

assuming ψ is unique:
if $\gamma_n > 0$ for some n then $\psi(\{0\}) = 0$ and

$$\int_{(0,\infty)} x^{-1}\psi(dx) < \infty$$

if $\gamma_n \equiv 0$ then

$$\psi(\{0\}) = \left\{\sum_n \frac{\alpha_1 \ldots \alpha_n}{\beta_2 \ldots \beta_{n+1}}\right\}^{-1} \geq 0$$

summary OPS on $[0, \infty)$

$\{P_n(x),\ n = 0, 1, \dots\}$ satisfies

$$P_n(x) = (x - \alpha_n - \beta_n - \gamma_n)P_{n-1}(x) - \alpha_{n-1}\beta_n P_{n-2}(x)$$
$$P_0(x) = 1, \quad P_1(x) = x - \alpha_1 - \gamma_1$$

with $\alpha_n > 0$, $\beta_{n+1} > 0$ and $\gamma_n \geq 0$

$\Longrightarrow$

$\{P_n(x),\ n = 0, 1, \dots\}$ is OPS with respect to measure ψ with support in $[0, \infty)$ and

$$\xi_1 = \lim_{n\to\infty} x_{n1} = \inf \operatorname{supp}(\psi)$$

$$\xi_i = \lim_{n\to\infty} x_{ni} = \inf \left\{\operatorname{supp}(\psi) \setminus \cup_{j<i}\, \xi_j\right\}$$

$$\sigma = \lim_{i\to\infty} \xi_i = \inf \overline{\operatorname{supp}(\psi)}$$

birth-death process with killing

definition: *birth-death process with killing* is Markov process $\{X(t),\ t \geq 0\}$ on $\{0, 1, \ldots\}$ with coffin state 0, *birth rate* $\alpha_n > 0$ and *killing rate* $\gamma_n \geq 0$ in state $n \geq 1$, and *death rate* $\beta_n > 0$ in state $n > 1$

representation for $i, j > 0$:

$$p_{ij}(t) := \Pr\{X(t) = j \,|\, X(0) = i\} = \pi_j \int_0^\infty e^{-xt} Q_i(x) Q_j(x) \psi(dx)$$

with

$$\pi_1 := 1,\ \pi_n := \frac{\alpha_1 \ldots \alpha_{n-1}}{\beta_2 \ldots \beta_n}\ (n > 1)$$

and

$$\alpha_n Q_n(x) = (\alpha_n + \beta_n + \gamma_n - x) Q_{n-1}(x) - \beta_n Q_{n-2}(x)$$
$$Q_0(x) = 1, \quad \alpha_1 Q_1(x) = \alpha_1 + \gamma_1 - x$$

birth-death processes with killing

$$p_{ij}(t) = \pi_j \int_0^\infty e^{-xt} Q_i(x) Q_j(x) \psi(dx)$$

$t = 0$:
$$\delta_{ij} = \pi_j \int_0^\infty Q_i(x) Q_j(x) \psi(dx)$$

defining

$$P_n(x) = (-1)^n \alpha_1 \alpha_2 \ldots \alpha_n Q_n(x)$$

we have

$$P_n(x) = (x - \alpha_n - \beta_n - \gamma_n) P_{n-1}(x) - \alpha_{n-1} \beta_n P_{n-2}(x)$$
$$P_0(x) = 1, \quad P_1(x) = x - \alpha_1 - \gamma_1$$

OPS with respect to measure ψ on $[0, \infty)$!

zeros of orthogonal polynomials

note: let

$$R_n = \begin{pmatrix} 0 & 0 & 0 & \cdots & \cdots \\ -\alpha_1 - \gamma_1 & \alpha_1 & 0 & \cdots & \cdots \\ \beta_2 & -\alpha_2 - \beta_2 - \gamma_2 & \alpha_2 & \cdots & \cdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \cdots & \cdots & \cdots & \cdots & \alpha_{n-1} \\ \cdots & \cdots & \cdots & \beta_n & -\alpha_n - \beta_n - \gamma_n \end{pmatrix}$$

truncated q-matrix of birth-death process with killing, then

$$P_n(x) = \det(xI_n + R_n)$$

so zeros of $P_n(x)$ are eigenvalues of $-R_n$

birth-death processes: decay rate

$$p_{ij}(t) = \pi_j \int_0^\infty e^{-xt} Q_i(x) Q_j(x) \psi(dx)$$

hence

$$p_j := \lim_{t\to\infty} p_{ij}(t) = \pi_j \psi(\{0\})$$

$$p_{ij}(t) - p_j = \pi_j \int_{0+}^\infty e^{-xt} Q_i(x) Q_j(x) \psi(dx)$$

interest: *decay rate*

$$\delta = \xi_1 + \xi_2 \mathbb{I}_{\{\xi_1 = 0\}}$$

note: if $\psi(\{0\}) = 0$ then $\xi_1 > 0$ or $\xi_1 = \xi_2 = 0$, so that $\delta = \xi_1$; if $\psi(\{0\}) > 0$ then $\xi_1 = 0$, so that $\delta = \xi_2$

birth-death processes: decay rate

given: birth rates α_n, death rates β_{n+1} and killing rates γ_n, $n \geq 1$

problem: determine decay rate $\delta = \xi_1 + \xi_2 \mathbb{I}_{\{\xi_1 = 0\}}$

recall: if $\gamma_n \equiv 0$ then $\psi(\{0\}) \geq 0$ is known; if $\gamma_n > 0$ for some n then $\psi(\{0\}) = 0$

if $\psi(\{0\}) = 0$ (and hence $\delta = \xi_1$):

Q1: $\xi_1 =$?

Q2: $\xi_1 > 0$?

if $\gamma_n \equiv 0$ and $\psi(\{0\}) > 0$ (and hence $\xi_1 = 0$ and $\delta = \xi_2$):

Q3: $\xi_2 =$? (*spectral gap*)

Q4: $\xi_2 > 0$?

birth-death processes: decay rate

problem: determine $\delta = \xi_1 + \xi_2 \mathbb{I}_{\{\xi_1=0\}}$

approach if $\psi(\{0\}) = 0$ (hence $\delta = \xi_1$): representations for ξ_1

approach if $\gamma_n \equiv 0$ and $\psi(\{0\}) > 0$ (hence $\delta = \xi_2$): *dual* process

definition: given $\alpha_n,\ \beta_{n+1}$ and $\gamma_n \equiv 0$ the dual process has rates

$$\tilde{\alpha}_n := \beta_{n+1}, \quad \tilde{\beta}_{n+1} := \alpha_{n+1}, \quad \tilde{\gamma}_1 := \alpha_1,\ \tilde{\gamma}_n := 0\ (n > 1)$$

then $\{\tilde{P}_n(x)\}$ OPS w.r.t $\tilde{\psi}$:

$$\alpha_1 \tilde{\psi}([0,x]) = \int_0^x u\psi(du), \quad x \geq 0$$

hence

$$\xi_2 = \tilde{\xi}_1$$

birth-death processes: decay rate

recall: (Geršgorin discs)

$$\xi_1 = \max_{\boldsymbol{a}>\mathbf{0}} \inf_i \left\{ c_i - a_{i+1} - \frac{\lambda_i}{a_i} \right\}$$

where $c_1 = \alpha_1 + \gamma_1$ and

$$c_i = \alpha_i + \beta_i + \gamma_i, \quad \lambda_i = \alpha_{i-1}\beta_i \quad (i > 1)$$

so, if $\gamma_i \equiv 0$ and $\psi(\{0\}) > 0$ (hence $\delta = \xi_2$) then

$$\begin{aligned} \delta = \xi_2 = \tilde{\xi}_1 &= \max_{\boldsymbol{a}>\mathbf{0}} \inf_i \left\{ \tilde{\alpha}_i + \tilde{\beta}_i + \tilde{\gamma}_i - a_{i+1} - \frac{\tilde{\alpha}_{i-1}\tilde{\beta}_i}{a_i} \right\} \\ &= \max_{\boldsymbol{a}>\mathbf{0}} \inf_i \left\{ \alpha_i + \beta_{i+1} - a_{i+1} - \frac{\alpha_i\beta_i}{a_i} \right\} \end{aligned}$$

decay rate: more recent results

setting: $\gamma_n \equiv 0$ and $\psi(\{0\}) > 0$ (so that $\xi_1 = 0$ and $\delta = \xi_2$)

Geršgorin + duality:

$$\delta = \max_{\boldsymbol{a}>\mathbf{0}} \inf_i \left\{ \alpha_i + \beta_{i+1} - a_{i+1} - \frac{\alpha_i \beta_i}{a_i} \right\}$$

Granovsky & Zeifman (1997), Chen (2001):

$$\begin{aligned} \delta &= \max_{\boldsymbol{a}>\mathbf{0}} \inf_i \left\{ \alpha_i + \beta_{i+1} - a_{i+1} - \frac{\alpha_i \beta_i}{a_i} \right\} \\ &= \min_{\boldsymbol{a}>\mathbf{0}} \sup_i \left\{ \alpha_i + \beta_{i+1} - a_{i+1} - \frac{\alpha_i \beta_i}{a_i} \right\} \end{aligned}$$

where $\beta_1 = 0$, $\boldsymbol{a} = (a_1, a_2, \ldots)$

decay rate: more recent results

setting: $\gamma_n \equiv 0$ and $\psi(\{0\}) > 0$ (so that $\xi_1 = 0$ and $\delta = \xi_2$)

Miclo (1999), Chen (2000):

$$\delta > 0 \iff \sup_i \left\{ \left(\sum_{j \le i} \frac{1}{\alpha_j \pi_j} \right) \left(\sum_{j > i} \pi_j \right) \right\} < \infty$$

where

$$\pi_1 = 1, \ \pi_j = \frac{\alpha_1 \ldots \alpha_{j-1}}{\beta_2 \ldots \beta_j} \ (j > 1)$$

recall: $\psi(\{0\}) > 0 \iff \sum_j \pi_j < \infty$

implications for OP's

translation Miclo-Chen result: explicit criterion for positivity of spectral gap if ξ_1 is known

theorem: let

$$P_n(x) = (x - c_n)P_{n-1}(x) - \lambda_n P_{n-2}(x)$$
$$P_0(x) = 1, \quad P_1(x) = x - c_1$$

and suppose $\xi_1 > -\infty$ then, defining $\pi_1 = 1$, $\alpha_1 = c_1 - \xi_1$ and, for $n > 1$,

$$\beta_n = \lambda_n/\alpha_{n-1}, \ \alpha_n = c_n - \xi_1 - \beta_n$$
$$\pi_n = (\alpha_1 \dots \alpha_{n-1})/(\beta_2 \dots \beta_n)$$

we have

$$\xi_2 > \xi_1 \iff \sup_i \left\{ \left(\sum_{j \le i} \frac{1}{\alpha_j \pi_j} \right) \left(\sum_{j > i} \pi_j \right) \right\} < \infty$$

decay rate: more recent results

setting: $\gamma_n \equiv 0$ and $\psi(\{0\}) > 0$ (so that $\xi_1 = 0$ and $\delta = \xi_2$)

Geršgorin + duality:

$$\delta = \max_{\boldsymbol{a}>\mathbf{0}} \inf_i \left\{ \alpha_i + \beta_{i+1} - a_{i+1} - \frac{\alpha_i \beta_i}{a_i} \right\}$$

Granovsky & Zeifman (1997), Chen (2001):

$$\begin{aligned} \delta &= \max_{\boldsymbol{a}>\mathbf{0}} \inf_i \left\{ \alpha_i + \beta_{i+1} - a_{i+1} - \frac{\alpha_i \beta_i}{a_i} \right\} \\ &= \min_{\boldsymbol{a}>\mathbf{0}} \sup_i \left\{ \alpha_i + \beta_{i+1} - a_{i+1} - \frac{\alpha_i \beta_i}{a_i} \right\} \end{aligned}$$

where $\beta_1 = 0$, $\boldsymbol{a} = (a_1, a_2, \ldots)$

extreme zeros of orthogonal polynomials

Granovsky-Zeifman-Chen result suggestive of

theorem: let

$$P_n(x) = (x - c_n)P_{n-1}(x) - \lambda_n P_{n-2}(x)$$
$$P_0(x) = 1, \quad P_1(x) = x - c_1$$

then, not only

$$\xi_1 = \max_{\boldsymbol{a}>\mathbf{0}} \inf_i \left\{ c_i - a_{i+1} - \frac{\lambda_i}{a_i} \right\}$$

but also

$$\xi_1 = \min_{\boldsymbol{a}>\mathbf{0}} \sup_i \left\{ c_i - a_{i+1} - \frac{\lambda_i}{a_i} \right\}$$

extreme zeros of orthogonal polynomials

more generally:
Granovsky-Zeifman-Chen result suggestive of

theorem: let

$$P_n(x) = (x - c_n)P_{n-1}(x) - \lambda_n P_{n-2}(x)$$
$$P_0(x) = 1, \quad P_1(x) = x - c_1$$

then, not only

$$x_{n1} = \max_{\boldsymbol{a}>\mathbf{0}} \min_{i \le n} \left\{ c_i - a_{i+1} - \frac{\lambda_i}{a_i} \right\}$$

but also

$$x_{n1} = \min_{\boldsymbol{a}>\mathbf{0}} \max_{i \le n} \left\{ c_i - a_{i+1} - \frac{\lambda_i}{a_i} \right\}$$

extreme zeros of orthogonal polynomials

$\{P_n(x),\ n = 0, 1, \ldots\}$ is OPS satisfying

$$P_n(x) = (x - c_n)P_{n-1}(x) - \lambda_n P_{n-2}(x)$$
$$P_0(x) = 1, \quad P_1(x) = x - c_1$$

with $c_n \in \mathbb{R},\ \lambda_n > 0$

recall: zeros of $P_n(x)$ are eigenvalues of

$$T_n = \begin{pmatrix} c_1 & \lambda_2/a_2 & 0 & \cdots & \cdots \\ a_2 & c_2 & \lambda_3/a_3 & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \cdots & \cdots & \cdots & c_{n-1} & \lambda_n/a_n \\ \cdots & \cdots & 0 & a_n & c_n \end{pmatrix}$$

where $a_j > 0$

extreme zeros of orthogonal polynomials

theorem:

$$x_{nn} = \max_{\boldsymbol{a}>\mathbf{0}} \min_{i\le n} \left\{ c_i + a_{i+1} + \frac{\lambda_i}{a_i} \right\}$$

proof: Perron-Frobenius theory for positive matrices

$\tilde{T}_n := T_n + dI$ positive for d sufficiently large, corresponds to $\tilde{c}_n := c_n + d$, $\tilde{\lambda}_n := \lambda_n$, and has eigenvalues $\tilde{x}_{ni} = x_{ni} + d$

Collatz-Wielandt:

$$x_{nn} = \max_{\boldsymbol{x}>\mathbf{0}} \min_{i} \frac{(T_n\boldsymbol{x})_i}{x_i} = \min_{\boldsymbol{x}>\mathbf{0}} \max_{i} \frac{(T_n\boldsymbol{x})_i}{x_i}$$

corollary:

$$x_{n1} = \min_{\boldsymbol{a}>\mathbf{0}} \max_{i\le n} \left\{ c_i - a_{i+1} - \frac{\lambda_i}{a_i} \right\}$$