Limit theorems for metapopulation processes subject to catastrophes

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  - Species utilising successional habitat depend on habitat dynamics;
  - Some species appear to have a negative impact on their local habitat.

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  - Y is the number of occupied patches.

The model of the half-hour...

(Habitat dynamics driven by catastrophes.)

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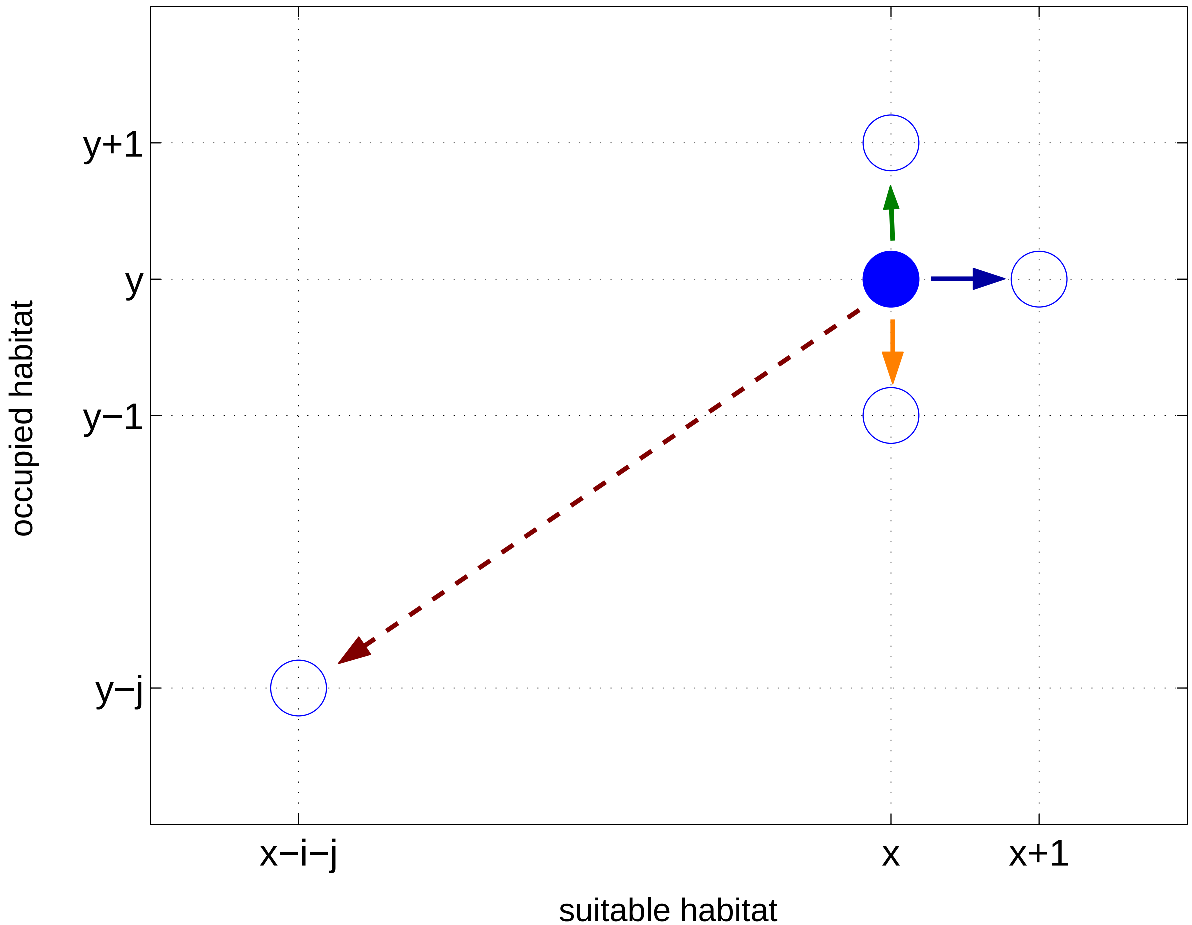

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- Each occupied patch produces migrants at rate c, which may colonise empty, suitable patches

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- Each local population goes extinct at rate e

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on $S = \{(x, y) \mid x, y \in \mathbb{N}, 0 \leq y \leq x \leq N\}$.

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$$(x, y) \to (x - (i + j), y - j) \quad \text{at rate}$$

$$\gamma \binom{x-y}{i} \binom{y}{j} p^{i+j} (1-p)^{x-i-j}.$$

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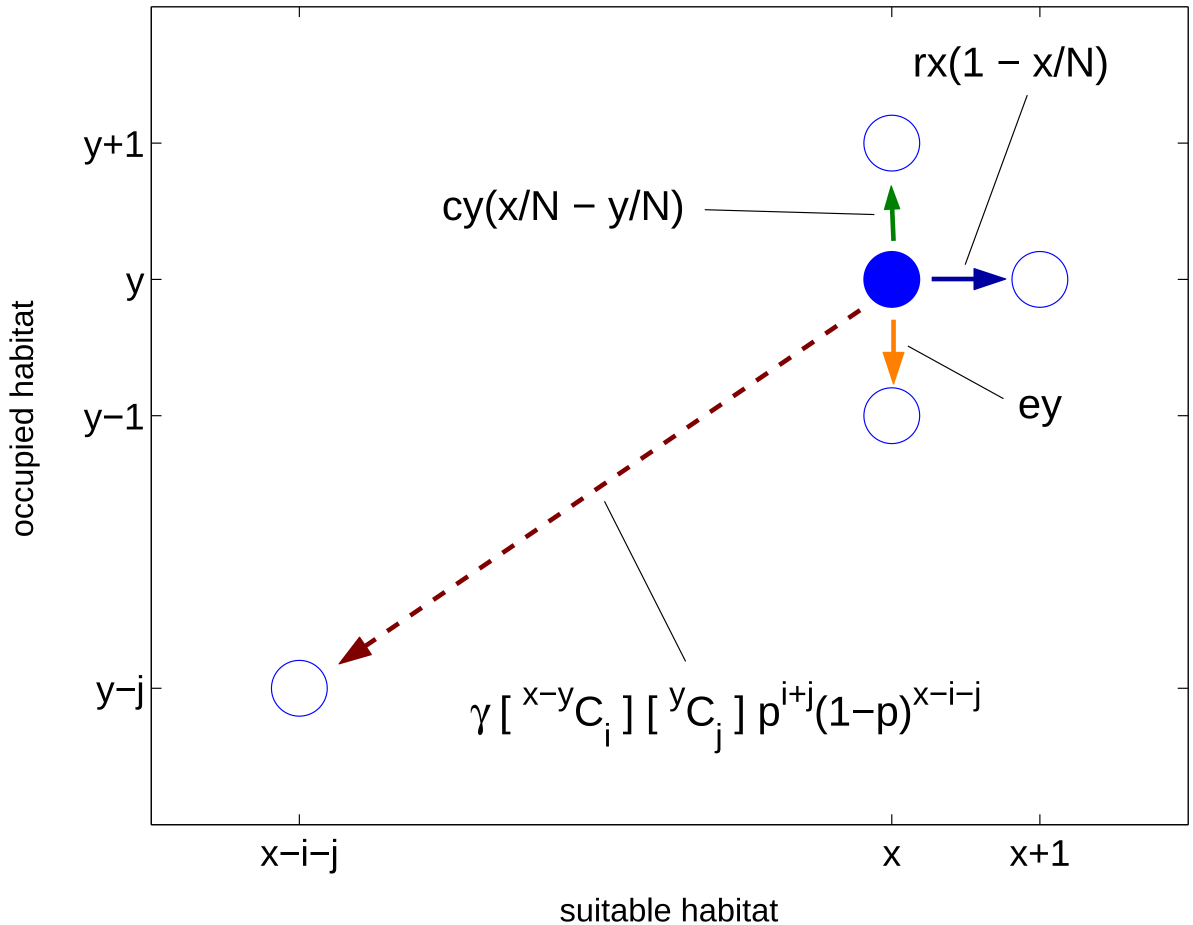
The model again...



Finite State-space Processes

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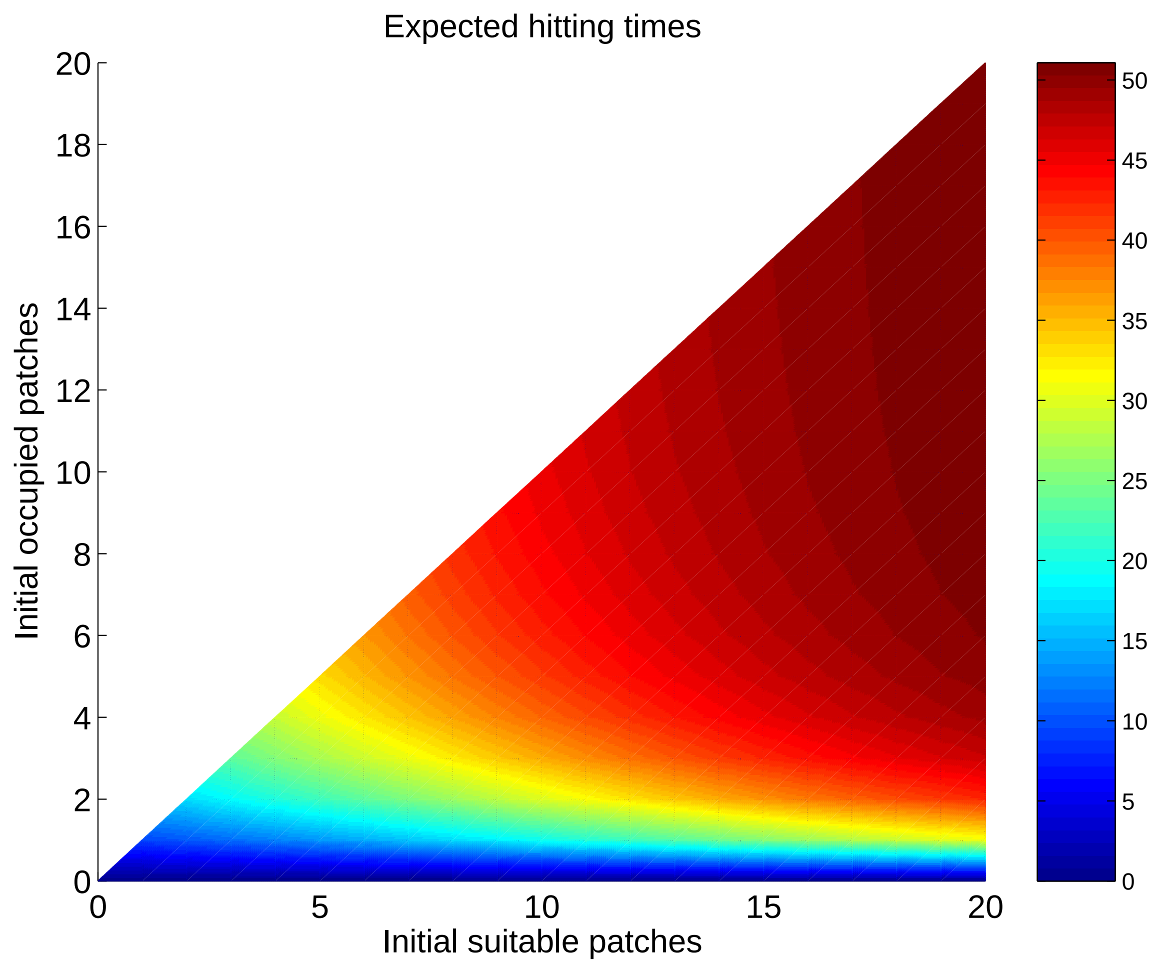
If N is small, (e.g.) expected extinction times are easy to calculate.

Extinction Times

$$Q_C \boldsymbol{\tau} = -1$$

Extinction Times



Difficulties as $N \uparrow$

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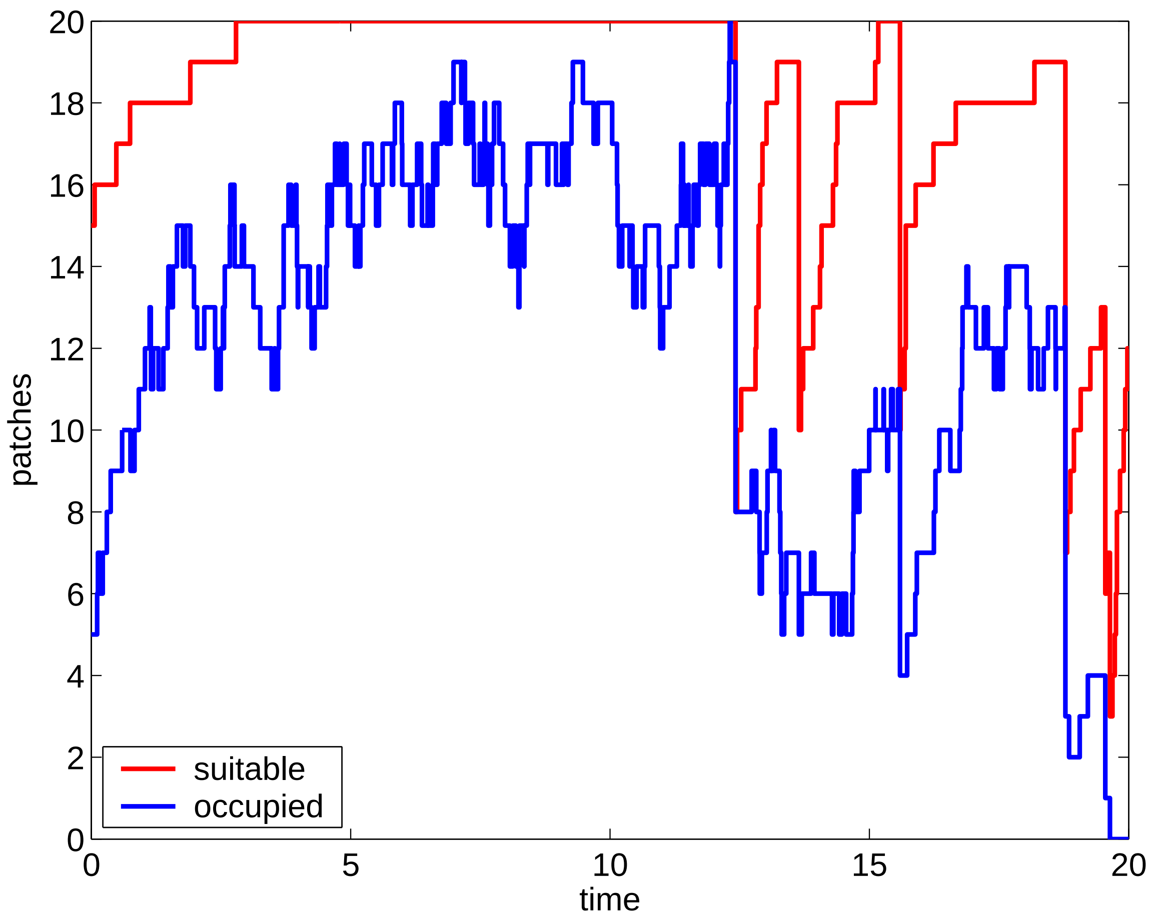
- To make progress, we need good approximations: *e.g.* stochastic differential equations for the limit as $N \to \infty$?

Simulations

Simulations can inform our intuition about the behaviour of a process, and suggest possible approximations.

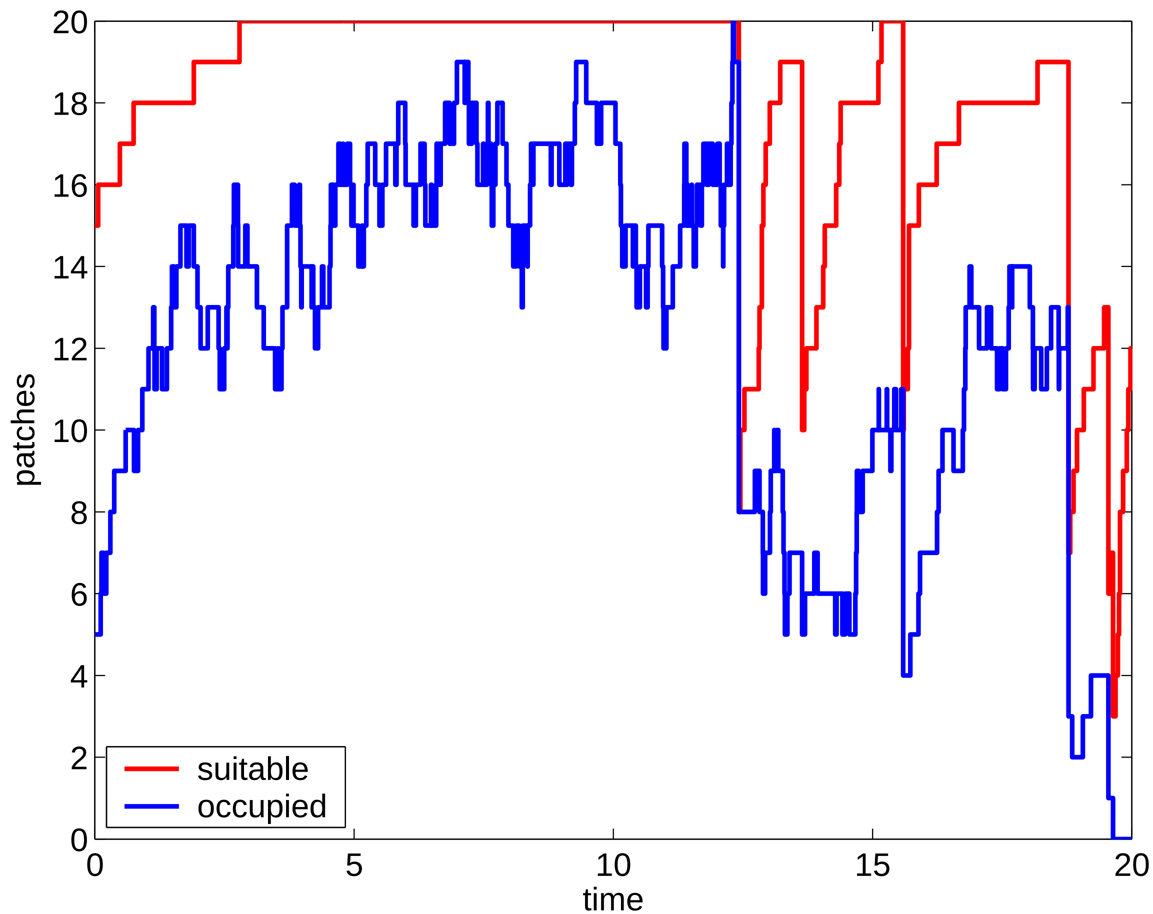
Simulations



Simulations

(How does the process change as N increases?)

Simulations



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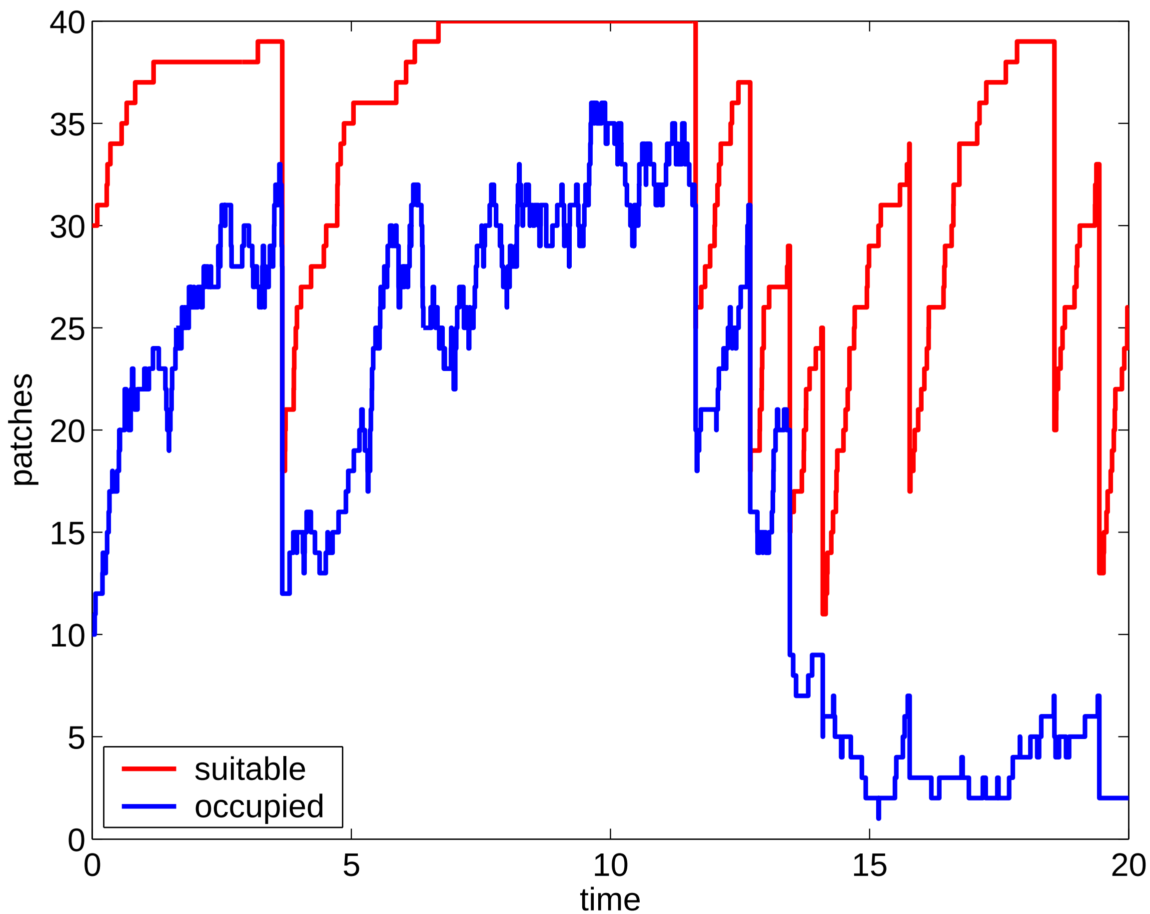

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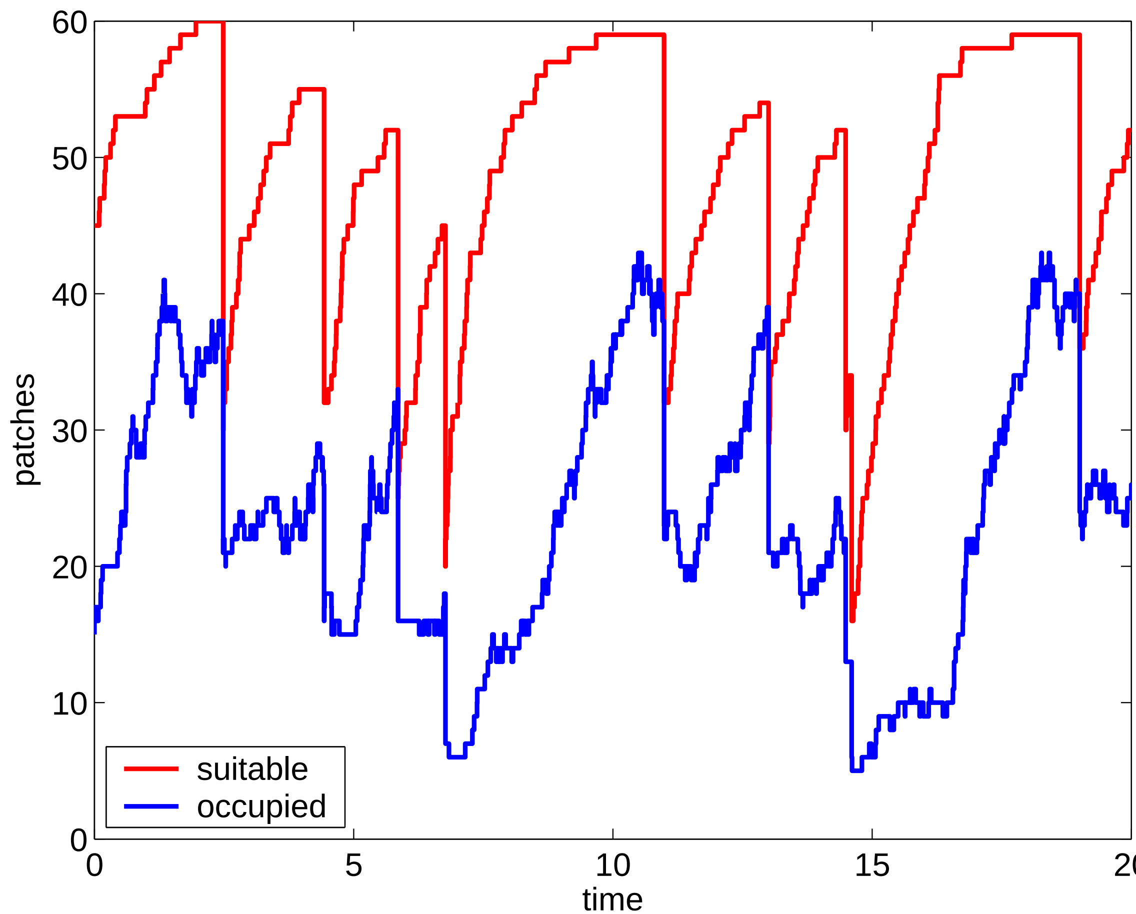

Simulations

As N increases, the process begins to look more deterministic.

Proportional scaling

In order to get a limit to the process, we need to scale it. If $\mathbf{X}(t) = [X(t)\ \ Y(t)]^T$ is our unscaled process, scale first by $1/N$.

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It is possible to show that as $N \to \infty$, $\left(N^{-1}\mathbf{X}_N - \hat{\mathbf{X}}_N\right) \Rightarrow \mathbf{0}$ (assuming $N^{-1}\mathbf{X}_N(0) = \hat{\mathbf{X}}_N(0)$).

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- $\hat{\mathrm{X}}_N$ is deterministic between catastrophes of fixed size, and these catastrophes occur at the same time as catastrophes in $\mathbf{X}_N$.

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& + \left[cy(x-y) - ey\right] f_y(x,y) \\
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(The generator tells us how the distribution of $\hat{\mathbf{X}}$ changes over time.)

Hitting times

There's a lot we can do with this simple scaling.

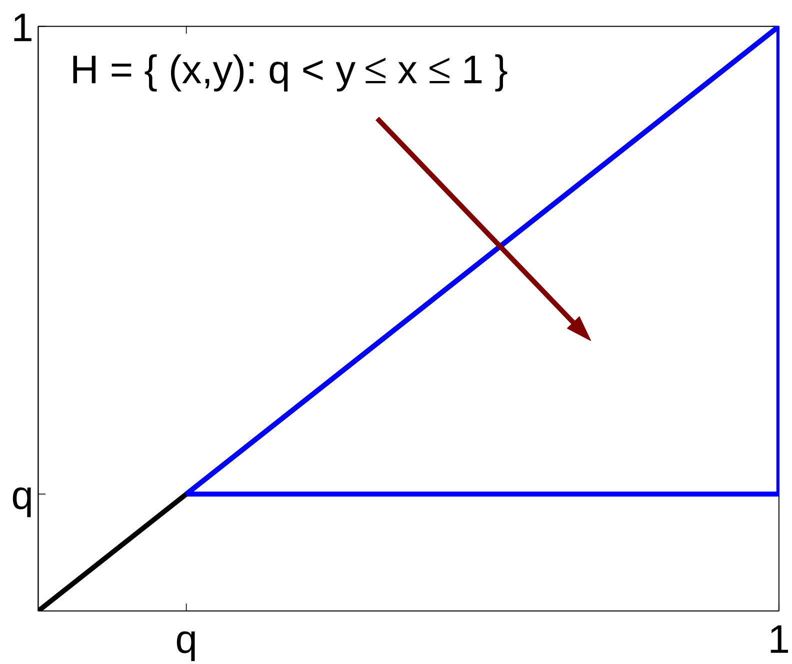
A solution $h(x,y)$ to the system of equations

$$G_{\hat{\mathbf{X}}} h(x,y) = -1, \quad (x,y) \in H \subset [0,1]^2,$$
$$h(x,y) = 0, \quad (x,y) \notin H,$$

gives the expected time to depart the set H (Gihman & Skorohod, 1972).

Hitting times

Example. Let $H = \{(x, y) : 0 \leq q < y \leq x \leq 1\}$, where q is a level of functional or quasi-extinction of the species, in terms of the proportion of occupied habitat patches.



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- The solution can be tough to evaluate (numerically)!
- The existence of a solution is not proven for other (even closely related) models.

A central limit scaling

Density dependent processes (as $\mathbf{X}_N$ would be, if not for the catastrophes) are well known to converge to Gaussian diffusion processes when appropriately scaled and normalised.

Let $Z_N(t) = \sqrt{N}\big(N^{-1}X_N(t) - \hat{\mathbf{X}}_N(t)\big)$. Then it is possible to show that

$$\begin{bmatrix} \mathbf{Z}_N \\ \hat{\mathbf{X}}_N \end{bmatrix} \Rightarrow \begin{bmatrix} \mathbf{Z} \\ \hat{\mathbf{X}} \end{bmatrix}.$$

A central limit

What is $\mathbb{Z}$? From the theory of density dependent processes we might expect a diffusion process with drift, plus catastrophes, and that is exactly what we get.

It is again possible to write down the generator...

Generator of Z

Generator $Gf(z_1, z_2, x_1, x_2)$ of $\left[\mathbf{Z}^T \ \hat{\mathbf{X}}^{\mathbf{T}}\right]^T$ has components:

- Drift components derived from
  $$\left[\frac{\partial F_i}{\partial z_j}\right] = \begin{bmatrix} -r & 0 \\ cz_2 & cz_1 - 2cz_2 - e \end{bmatrix}.$$
- Diffusion components $\frac{1}{2}r(1 - z_1)f_{z_1 z_1}$ and $\frac{1}{2}[cz_2(z_1 - z_2) + ez_2]f_{z_2 z_2}$.
- Catastrophe component obtained by $\mathrm{E}[f(Z_1 + U_1, Z_2 + U_2, \cdot) - f(Z_1, Z_2, \cdot)]$, where (U_1, U_2) is bivariate normal.

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...and we could approximate $\mathbf{X}_N$ by $\hat{\mathbf{X}} + N^{-1/2}\mathbf{Z}$.

Future directions

- Extension of this approach to general density dependent processes subject to a wider class of catastrophes.
- Investigation of the accuracy of the approximations and their properties (e.g. hitting times).
- When do features of small-N processes remain in large-N approximations?

Thanks

- Phil Pollett and Hugh Possingham (advisors), Andrew Barbour and Chris Wilcox.

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