# Which Markov chains have a given invariant measure?

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# Fun at the Water Park

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It is called standard if

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and honest if

- $\sum_j p_{ij}(t) = 1$, for some (and then for all) $t > 0$.

# The $q$-matrix

For a standard process $P$, the right-hand derivative $p'_{ij}(0+) = q_{ij}$ exists and defines a $q$-matrix $Q = (q_{ij}, i, j \in S)$. Its entries satisfy

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We set $q_i = -q_{ii}$, $i \in S$.

Suppose that $Q$ is given. Assume that $Q$ is stable, that is $q_i < \infty$ for all $i$ in $S$. A standard process $P$ will then be called a $Q$-process if its $q$-matrix is $Q$.

# The Kolmogorov DEs

For simplicity, we assume $Q$ is conservative, that is

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Under this condition, every $Q$-process $P$ satisfies the backward equations,

$$\mathrm{BE}_{ij} \qquad p'_{ij}(t) = \textstyle\sum_k q_{ik} p_{kj}(t), \quad t > 0,$$

but might not satisfy the forward equations,

$$\mathrm{FE}_{ij} \qquad p'_{ij}(t) = \textstyle\sum_k p_{ik}(t) q_{kj}, \quad t > 0.$$

# Stationary distributions

A collection of positive numbers $\pi = (\pi_j, j \in S)$ is a stationary distribution if $\sum_j \pi_j = 1$ and

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**Recipe for finding a stationary distribution!**

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**Recipe.** Find a collection of strictly positive numbers $m = (m_j, j \in S)$ such that

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**Recipe.** Find a collection of strictly positive numbers $m = (m_j, j \in S)$ such that

$$\sum_i m_i q_{ij} = 0.$$

Such an $m$ is called an invariant measure for $Q$. If $\sum_i m_i < \infty$, we set $\pi_j = m_j / \sum_i m_i$ and hope $\pi$ satisfies (1).

# Birth-death processes

**Transition rates.**

$q_{i,i+1} = \lambda_i$ ($\uparrow$ - birth rates)

$q_{i,i-1} = \mu_i$ ($\downarrow$ - death rates) ($\mu_0 = 0$)

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**Solution.** $m_0 = 1$ and

$$m_j = \prod_{i=1}^{j} \frac{\lambda_{i-1}}{\mu_i}, \quad j \geq 1.$$

# Miller's example

**Transition rates.** Fix $r > 0$ and set

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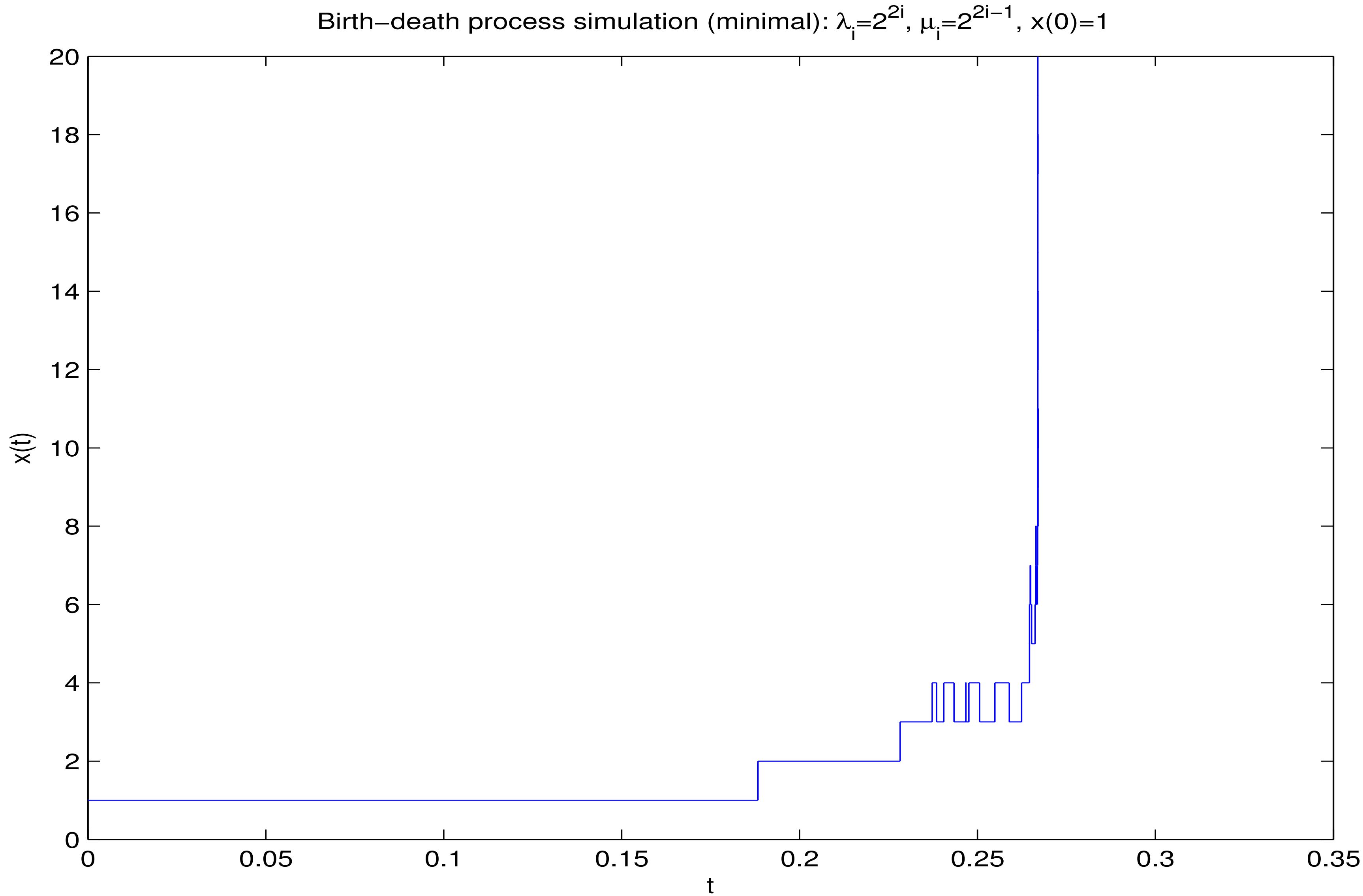
$$m_j = \prod_{i=1}^{j} \frac{\lambda_{i-1}}{\mu_i}, \quad j \geq 1.$$

So, $m_j = \rho^j$, where $\rho = 1/r$, and hence if $r > 1$,

$$\pi_j = (1-\rho)\rho^j, \quad j \geq 0.$$

# Simulation



Birth−death process simulation (minimal): $\lambda_i=2^{2i}$, $\mu_i=2^{2i-1}$, x(0)=1
x(t)
t
0
2
4
6
8
10
12
14
16
18
20
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35

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The relative proportion of births to deaths is $r$ and so, if $r > 1$, the "process" is clearly transient.

In fact, the "process" is explosive. ($Q$ is not regular.) R.G. Miller* showed that $Q$ needs to be regular for the recipe to work.

*Miller, R.G. Jr. (1963) Stationary equations in continuous time Markov chains. *Trans. Amer. Math. Soc.* 109, 35–44.

# Motivating question

If $Q$ is regular, then there exists uniquely a $Q$-process, namely the minimal process: the minimal solution $F(\cdot) = (f_{ij}(\cdot), i, j \in S)$ to BE$_{ij}$.

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Question. Suppose that there exists a collection of strictly positive numbers $\pi = (\pi_j, j \in S)$ such that

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Does $\pi$ admit an interpretation as a stationary distribution for any of these processes?

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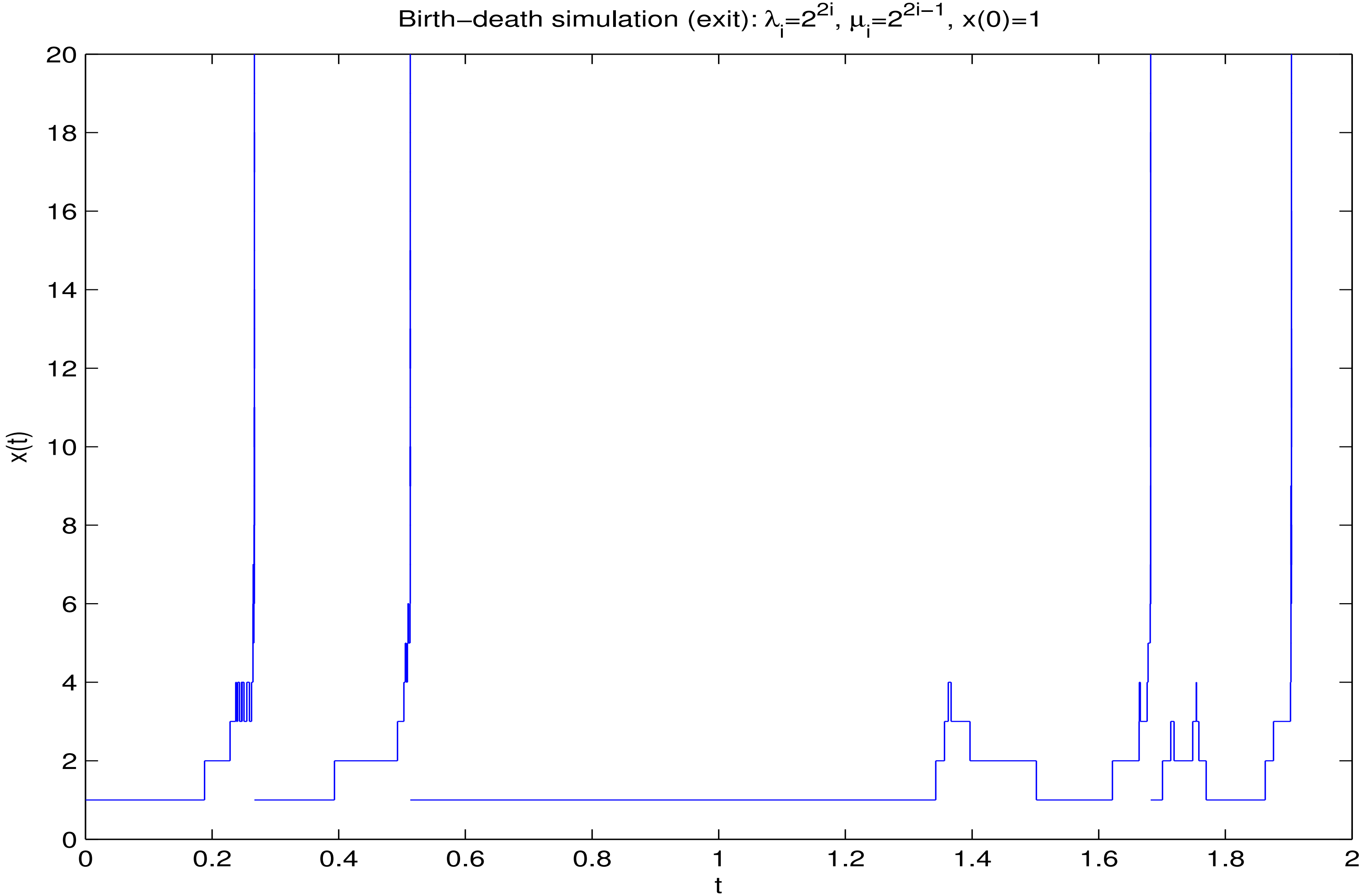

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1
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2

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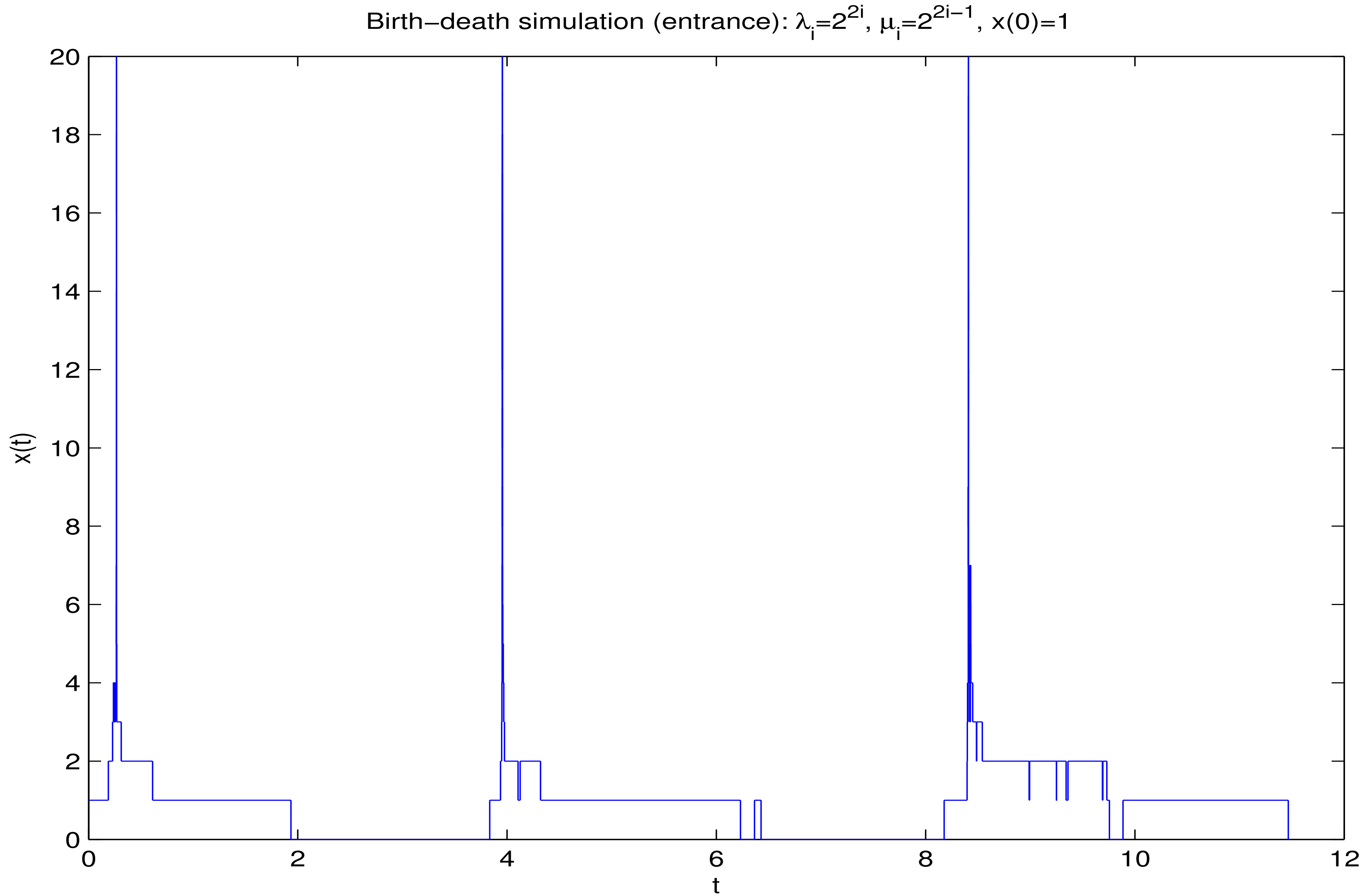

Birth−death simulation (entrance): $\lambda_i=2^{2i}$, $\mu_i=2^{2i-1}$, x(0)=1
x(t)
t
0
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6
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**Theorem.** Let $P$ be an arbitrary $Q$-process. If $m$ is invariant for $P$, then $m$ is subinvariant for $Q$:

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**Theorem.** Let $P$ be an arbitrary $Q$-process. If $m$ is invariant for $P$, then $m$ is subinvariant for $Q$, and invariant for $Q$ if and only if $P$ satisfies the forward equations $\mathrm{FE}_{ij}$ over $S$:

$$\left( \sum_i m_i p_{ij}(t) = m_j \quad \Rightarrow \quad \sum_i m_i q_{ij} = 0 \right) \quad \Leftrightarrow \quad \mathrm{FE}$$

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**Corollary.** If $m$ is invariant for the minimal process $F$, then $m$ is invariant for $Q$.

# A construction problem

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**Problem 3.** When such a $Q$-process exists, is it unique?

**Problem 4.** In the case of non-uniqueness, can one identify all $Q$-processes (or perhaps all honest $Q$-processes) for which $m$ is invariant?

# The resolvent

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$$\psi_{ij}(\lambda) = \int_0^\infty e^{-\lambda t} p_{ij}(t)dt, \quad \lambda > 0,$$

for the Laplace transform of $p_{ij}(\cdot)$, then $\Psi(\cdot) = (\psi_{ij}(\cdot), i, j \in S)$ enjoys the following properties:

- $\psi_{ij}(\lambda) \geq 0$, $\sum_j \lambda\psi_{ij}(\lambda) \leq 1$, and
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$\Psi$ is called the resolvent of $P$. Indeed, if $\Psi$ is a given resolvent, in that it satisfies these properties, then there exists a standard (!) process $P$ with $\Psi$ as its resolvent*.

* Reuter, G.E.H. (1967) Note on resolvents of denumerable submarkovian processes. *Z. Wahrscheinlichkeitstheorie* 9, 16–19.

# Identifying $Q$-processes

Now, if one is given a stable and conservative $q$-matrix $Q$, and a resolvent $\Psi$ satisfying the backward equations,

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One can also use the resolvent to determine whether or not the $Q$-process is honest. This happens if and only if

$$\sum_j \lambda\psi_{ij}(\lambda) = 1, \quad i \in S,\ \lambda > 0.$$

# Identifying invariant measures

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**Theorem.** Let $P$ be an arbitrary process and let $\Psi$ be its resolvent. Then, $m$ is invariant for $P$ if and only if it is invariant for $\Psi$, that is,

$$\sum_i m_i p_{ij}(t) = m_j$$

if and only if

$$\sum_i m_i \lambda \psi_{ij}(\lambda) = m_j.$$

# Steps to identifying $P$

Steps to identifying a $Q$-process (an honest $Q$-process) for which a given $m$ is invariant:

- $\psi_{ij}(\lambda) \geq 0$, $\sum_j \lambda \psi_{ij}(\lambda) \leq 1$, and
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Then, if $d = 0$, $m$ is invariant for the minimal $Q$-process. Otherwise, if $\sum_i d_i(\lambda) \le \sum_i m_i z_i(\lambda) < \infty$, for all $\lambda > 0$, there exists a $Q$-process $P$ for which $m$ is invariant.

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**Theorem continued.** The resolvent of one such process is given by

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + \frac{z_i(\lambda) d_j(\lambda)}{\lambda \sum_k m_k z_k(\lambda)}, \tag{2}$$

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**Corollary.** If $m$ is a subinvariant probability distribution for $Q$, then there exists an honest $Q$-process with stationary distribution $m$. The resolvent of one such process is given by (2).

# The single-exit case

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is necessary for the existence of a $Q$-process for which the specified measure is invariant; the $Q$-process is then determined uniquely by

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + \frac{z_i(\lambda)d_j(\lambda)}{\lambda\sum_k m_k z_k(\lambda)}.$$

# Non-uniqueness

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Consider a pure-birth process with strictly positive birth rates $(q_i, i \geq 0)$, but imagine that we have two distinct sets of birth rates, $(q_i^{(0)}, i \geq 0)$ and $(q_i^{(1)}, i \geq 0)$, which satisfy $\sum_{i=0}^{\infty} 1/q_i^{(r)} < \infty$, $r = 0, 1$.

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$$q_{(r,i)(s,j)} = \begin{cases} q_i^{(r)}, & \text{if } j = i + 1 \text{ and } s = r, \\ -q_i^{(r)}, & \text{if } j = i \text{ and } s = r, \\ 0, & \text{otherwise,} \end{cases}$$

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for $r = 0, 1$ and $i \geq 0$. The measure $m = (m_x, x \in S)$, given by $m_{(r,i)} = 1/q_i^{(r)}$, $r = 0, 1,\ i \geq 0$, is subinvariant for $Q$.

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**Corollary**. If $Q$ is reversible with respect to $m$, then there exists uniquely a $Q$-function $P$ for which $m$ is invariant if and only if $\sum_j m_j z_j(\lambda) < \infty$, for all $\lambda > 0$.

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**Corollary.** If $Q$ is reversible with respect to $m$, then there exists uniquely a $Q$-function $P$ for which $m$ is invariant if and only if $\sum_j m_j z_j(\lambda) < \infty$, for all $\lambda > 0$. It is honest and its resolvent is given by

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Moreover, $P$ is reversible with respect to $m$ in that $m_i p_{ij}(t) = m_j p_{ji}(t)$ (or, equivalently, $m_i \psi_{ij}(\lambda) = m_j \psi_{ji}(\lambda)$).

*Hou Chen-Ting and Chen Mufa (1980) Markov processes and field theory. *Kexue. Tongbao* 25, 807–811.

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# $\mu$-Invariance

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It is called a $\mu$-invariant measure for $P$, where $P$ is any transition function, if

$$\sum_{i \in C} m_i p_{ij}(t) = e^{-\mu t} m_j, \quad j \in C.$$

# Quasi-stationary distributions

**Proposition.** A probability distribution $\pi = (\pi_i, i \in C)$ is a $\mu$-invariant measure for some $\mu > 0$, that is,

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if and only if it is a quasi-stationary distribution: for $j \in C$,

$$p_j(t) = \sum_{i \in C} m_i p_{ij}(t) \Rightarrow \frac{p_j(t)}{\sum_{k \in C} p_k(t)} = m_j.$$

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**Theorem.** If $m$ is $\mu$-invariant for $Q$, then it is $\mu$-invariant for $F$ if and only if the equations $\sum_{i \in C} y_i q_{ij} = -\nu y_j$, $0 \leq y_i \leq m_i$, $i \in C$, have no non-trivial solution for some (and then all) $\nu < \mu$.

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**Theorem.** If $m$ is a finite $\mu$-invariant measure for $Q$, then

$$\mu \sum_{i\in C} m_i a_i^F \le \sum_{i\in C} m_i q_{i0}, \tag{4}$$

where $a_i^F = \lim_{t\to\infty} f_{i0}(t)$, and $m$ is $\mu$-invariant for $F$ if and only if equality holds in (4).

# $Q$-processes with a given $m$

**Theorem.** (Existence) Let $\mu > 0$ and suppose that $Q$ admits a finite $\mu$-subinvariant measure $m$ on $C$.

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**Theorem.** (Existence) Let $\mu > 0$ and suppose that $Q$ admits a finite $\mu$-subinvariant measure $m$ on $C$.

1. If the minimal $Q$-process $F$ is honest, then $m$ is a $\mu$-invariant measure on $C$ for $F$ if and only if

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2. If $F$ is dishonest, then there exists a $Q$-process $P$ for which $m$ is $\mu$-invariant on $C$ if and only if

$$\sum_{i \in C} m_i q_{i0} \le \mu \sum_{i \in C} m_i.$$

# $Q$-processes with a given $m$

**Theorem continued.**

The resolvent $\Psi$ of one such $Q$-process for which $m$ is $\mu$-invariant has the form

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + \frac{z_i(\lambda) d_j(\lambda)}{(\lambda + \mu) \sum_{k \in C} m_k z_k(\lambda)}, \qquad i, j \in S,$$

where $d_j(\lambda) = m_j - \sum_{i \in C} m_i (\lambda + \mu) \phi_{ij}(\lambda)$, $\quad j \in C$,

$$d_0(\lambda) = e/\lambda - \sum_{i \in C} m_i (\lambda + \mu) \phi_{i0}(\lambda),$$

and $e$ satisfies $\sum_{i \in C} m_i q_{i0} \le e \le \mu \sum_{i \in C} m_i$.

# $Q$-processes with a given $m$

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1. If $m$ is $\mu$-invariant for the minimal $Q$-process $F$, which is true if and only if $\mu \sum_{i \in C} m_i a_i^F = \sum_{i \in C} m_i q_{i0}$, then it is the unique $Q$-process for which $m$ is $\mu$-invariant on $C$. When this condition holds, $m$ is $\mu$-invariant on $C$ for $Q$.

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$$\sum_{i \in C} m_i q_{i0} \le \mu \sum_{i \in C} m_i. \tag{5}$$
3. If $Q$ is single-exit, there exists uniquely $Q$-process for which $m$ is $\mu$-invariant if and only if (5) holds.

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**Theorem continued.** If If $Q$ is single-exit, and $\sum_{i \in C} m_i q_{i0} \leq \mu \sum_{i \in C} m_i$ then all $Q$-processes for which $m$ is $\mu$-invariant can be constructed using

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + \frac{z_i(\lambda) d_j(\lambda)}{(\lambda+\mu) \sum_{k \in C} m_k z_k(\lambda)}, \qquad i, j \in S,$$

where $d_j(\lambda) = m_j - \sum_{i \in C} m_i (\lambda + \mu) \phi_{ij}(\lambda), \quad j \in C$,

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