Assignment Number 4

**Problem 1** Prove Minkowski's inequality: for $1 \leq p \leq \infty$ and $u, v \in L^p(\Omega)$ ($\Omega$ a domain in $\mathbb{R}^n$, $n \geq 1$), there holds:

$$||u + v||_{L^p(\Omega)} \leq ||u||_{L^p(\Omega)} + ||v||_{L^p(\Omega)}.$$ 

**Problem 2** Let $B$ be the open unit ball in $\mathbb{R}^n$, $n > 1$. Show that the following function is unbounded, but lies in $W^{1,n}(B)$:

$$x \mapsto \log(\log(1 + \frac{1}{|x|})).$$

**Problem 3** Let $X$ be a topological space, and set $I := [0,1]$. The inverse of a path $f$ in $X$ (notation as in Assignment 2) is defined by

$$f^{-1}(t) = f(1 - t) \quad t \in I.$$ 

The constant path in $p \in X$ is defined via $i_p(t) = p$ for all $t \in I$.

a) Does there always hold $f \ast f^{-1} = f^{-1} \ast f$?
b) Let $f$ be a path in $X$ with $f(0) = f(1) = p$, (a loop with basepoint $p$). Does there hold $f \ast f^{-1} = f^{-1} \ast f = i_p$?
c) The path class of a path $f$, $[f]$, is the equivalence class of $f$ w.r.t. $\simeq_0 I$. In particular, $[f](0), [f](1)$ are well defined.

Fix $p \in X$. Show that the set

$$\{[f] : [f]$ is a path class with $[f](0) = [f](1) = p\}$$

is a group with the binary operation $[f][g] := [f \ast g]$, identity $[i_p]$ and $[f]^{-1} := [f^{-1}]$. The group is called the fundamental group of $x$ with basepoint $p$, and is denoted $\pi_1(X,p)$.

Due: Thursday, 5/5/2005 before the tutorial

Current assignments will be available at