Assignment Number 3

Problem 1 Let \((X, d)\) be a complete metric space, and let \(T : X \to X\) be a contraction mapping, i.e., there exists \(\theta \in [0, 1)\) with

\[
d(T(x), T(y)) \leq \theta d(x, y) \quad \text{for all } (x, y \in X).
\]

(Note: \(T\) is not necessarily linear.) Show that there exists precisely one \(x \in X\) with \(T(x) = x\). Does the statement remain true if the completeness requirement is omitted?

Problem 2 a) For \(p = 1, \frac{2}{3}, 2, 3, 10, \infty\), sketch the “unit \(p\)-ball” w.r.t. \(| \cdot |_p\) in \(\mathbb{R}^2\).

b) Let \(x \mapsto |x| \) and \(x \mapsto |x|'\) be two norms on a \(K\)-vectorspace \(V\), \(K = \mathbb{R}\) or \(K = \mathbb{C}\). Show the equivalence of the following statements:

(i) There exist \(c\) and \(C\) with \(0 < c < C < \infty\), such that: \(c|x| \leq |x|' \leq C|x|\).

(ii) In every open \(| \cdot |\)-ball there is a \(| \cdot |'\)-ball, and vice versa.

(iii) The topologies induced by \(| \cdot |\) and \(| \cdot |'\) coincide.

(iv) A sequence \(\{x_i\}\) converges to a point \(x\) w.r.t. \(| \cdot |\) iff \(\{x_i\}\) converges to \(x\) w.r.t. \(| \cdot |'\).

Problem 3 Let \(\Omega \subseteq \mathbb{R}^n\) be open and bounded (and nonempty), with smooth boundary, and let \(\alpha \in (0, 1]\) be fixed. A function \(f : \Omega \to \mathbb{R}\) is called uniformly Hölder continuous on \(\Omega\) (with exponent \(\alpha\)), if there exists \(K < \infty\) with

\[
\sup_{x, y \in \Omega, \ x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha} \leq K \quad (\ast).
\]

The space of all such functions is denoted by \(C^\alpha(\overline{\Omega})\). If(\ast) only holds on every compact subset \(\Omega' \subseteq \Omega\), then \(f\) is called locally Hölder continuous on \(\Omega\) (with exponent \(\alpha\)), denoted \(f \in C^\alpha(\Omega)\); note that here, \(K\) may depend on \(\Omega'\).

a) Show: \(C^\alpha(\overline{\Omega}) \subseteq C^0(\overline{\Omega})\). Show by virtue of an example that the inclusion is strict.

b) Prove: \(C^\alpha(\overline{\Omega}) \subset C^\beta(\overline{\Omega})\), but \(C^\alpha(\overline{\Omega}) \neq C^\beta(\overline{\Omega})\) for \(0 < \beta < \alpha \leq 1\). (The analogous statements hold for \(C^\alpha(\Omega)\) and \(C^\beta(\Omega)\)).

c) We define:

\[
[f]_{\Omega, \alpha} := \sup_{x, y \in \Omega, \ x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.
\]

Is \(| \cdot |_{\Omega, \alpha}\) a norm on \(C^\alpha(\overline{\Omega})\)? A seminorm?

Due: Thursday, 21/4/2005 before the tutorial

Current assignments will be available at