Assignment Number 2

Problem 1 Let $X$ be a metric space. Show that there holds:

a) $|d(x, z) - d(y, z)| \leq d(x, y)$ (the reverse triangle inequality).
b) $\tilde{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)}$ defines a metric on $X$ which is equivalent to $d$ (i.e., $\tilde{d}$ is a metric on $X$, and $x_i \to x$ with respect to $d$ is equivalent to $x_i \to x$ with respect to $\tilde{d}$). What happens if $d$ is an extended metric?

Problem 2 Let $(X, \tau)$ be a compact topological space. Show:

a) If $Y$ is Hausdorff and $f : X \to Y$ is continuous, then $f(X)$ is a compact subset of $Y$.
b) If $f : X \to \mathbb{R}$ is continuous, then there exists $x_0 \in X$ with $f(x) \leq f(x_0)$ for every $x \in X$ (i.e., $f$ attains its maximum).
c) If $Y$ is Hausdorff and $f : X \to Y$ is continuous and bijective, then $f^{-1}$ is continuous (and hence $f$ is a homeomorphism).

Problem 3 Let $X$ be the set of all functions $f : [0, 1] \to \mathbb{R}^n$ which satisfy $f(0) = 0$ and $|f(x) - f(y)| \leq |x - y|$. Given $\varphi \in C(\mathbb{R}^n, \mathbb{R})$, define $\Phi : X \to \mathbb{R}$ via

$$\Phi(f) := \int_0^1 \varphi(f(t)) \, dt.$$ 

Show:

a) $X$ is a compact subset of $C([0, 1], \mathbb{R}^n)$ (Hint: Arzela-Ascoli).
b) $\Phi : X \to \mathbb{R}$ is continuous.
c) Use Problem 2b) to show that there exists at least one function $f_0 \in X$ with $\Phi(f) \geq \Phi(f_0)$ for every $f \in X$.

Due: Thursday, 7/4/2004 before the tutorial

Current assignments will be available at