Assignment Number 1

Problem 1 Let $X$ be a topological space, $A \subseteq X$. Show that there holds:

a) $A$ is open iff $\text{Int} A = A$;

b) $A$ is closed iff $\overline{A} = A$.

Problem 2 Let $X$ and $Y$ be topological spaces, $I = [0, 1]$. Functions $f, g \in C(X, Y)$ are called homotopic (i.e., $f \simeq g$) if there exists a homotopy between $f$ and $g$, i.e., $F \in C(I \times X, Y)$ with $F(0, \cdot) = f(\cdot)$, $F(1, \cdot) = g(\cdot)$.

a) Show that $\simeq$ is an equivalence relationship on $C(X, Y)$, the space of continuous maps from $X$ to $Y$.

b) Let $A \subseteq X$. Maps $f$ and $g$ in $C(X, Y)$ with $f \big|_A = g \big|_A$ are called homotopic relative to $A$ (written: $f \simeq_A g$, or $f \simeq g$ rel $A$), if there is a homotopy $F$ between $f$ and $g$ that further satisfies: $F(\cdot, a) = f(a) (= g(a))$ for every $a \in A$. Show that $\simeq_A$ is also an equivalence relation on $C(X, Y)$.

c) Now consider $X = I$, $A = \partial I$. A path in $Y$ is a map in $C(I, Y)$. The composition of two paths from $f$ and $g$ in $Y$ is defined by:

$$ (f * g)(t) = \begin{cases} f(2t) & \text{for } 0 \leq t < \frac{1}{2} \\ g(2t - 1) & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases} $$

The composition $f * g$ is obviously a path iff $f(1) = g(0)$. Now consider paths $f_1$, $f_2$, $g_1$ and $g_2$ in $Y$ with $f_1 \simeq_{\partial I} f_2$, $g_1 \simeq_{\partial I} g_2$. Show that $f_1(1) = g_1(0)$ implies: $f_1 * g_1 \simeq_{\partial I} f_2 * g_2$.

Problem 3 Let $(X, d)$ be a metric space, $A \subseteq X$. Let $d(x, A)$ denote the distance from a point $x \in X$ to $A$, i.e. $d(x, A) = \inf_{a \in A} d(x, a)$. Show:

$x$ is an accumulation point of $A \iff d(x, A \setminus \{x\}) = 0 \iff$ there exists a sequence $\{x_i\}$ in $A$ with limit $x$ and $x_i \neq x$ for all $i \in \mathbb{N}$.

Due: Thursday, 17/3/2004 before the tutorial

Current assignments will be available at