MATH4106/7106 Assignment 2 AJB/2004

(Solutions to starred questions due Friday, August 27, 5pm)

⋆ 1. The Legendre polynomial of \( n \)-th degree can be defined by

\[
P_n(x) = \frac{1}{\pi} \int_0^\pi [x + (x^2 - 1)^{1/2} \cos(\theta)]^n d\theta, \quad (x > 1),
\]

where the positive square root is taken. Find \( P_0(x) \), \( P_1(x) \), \( P_2(x) \). Use Laplace’s Method to deduce that

\[
P_n(x) \sim \frac{1}{\sqrt{2\pi n}} \frac{[x + (x^2 - 1)^{1/2}]^{n+1/2}}{(x^2 - 1)^{1/4}} \quad (n \to \infty).
\]

2. Find the leading behaviour of

\[
\int_0^1 \sin\left(x[t + \frac{t^3}{6} - \sinh(t)]\right) \cos(t) \, dt, \quad (x \to \infty).
\]

⋆ 3. Find and sketch the lines of greatest ascent/descent for the real part of \( \rho(t) = (t - 1)^2 \). Investigate the effects on the contours near \( t = 0 \) and \( t = 1 \), of adding

(a) \( \frac{1}{t} \)

(b) \( t^{1/2} \) (principal value)

to \( \rho(t) \).

⋆ 4. Let

\[
f(x) = \int_0^1 \frac{1}{\sqrt{t}} e^{ix(t+t^2)} \, dt, \quad x > 0,
\]

where the path of integration is along the real axis. Find suitable steepest descent/ascent contours through \( t = 0, t = 1 \) in the complex \( t \)-plane and show carefully that as \( x \to \infty \),

\[
f(x) \sim \sqrt{\frac{\pi}{x}} e^{ix/4} \left(1 - \frac{3i}{4x} - \frac{105}{32x^2} + \ldots\right) + i e^{2ix} \left(-\frac{1}{x} + \frac{7i}{18x^2} + \frac{111}{324x^3} + \ldots\right).
\]
5. Let
\[ \varphi(x, \xi) = \frac{1}{2\pi i} \int_{C_1} e^{i(x \xi - \sqrt{s})} ds \]
where \( \xi > 0 \) is constant, \( x > 0 \), and the path of integration is along a vertical line \( C_1 \) that is a distance \( \gamma \) to the right of the origin. The quantity \( \sqrt{s} \) is given its principal value. Show that the path of integration can be deformed to one passing through a saddle point of \( (s \xi - \sqrt{s}) \), and then use Laplace’s method to deduce that
\[ \varphi(x, \xi) \sim 2 \sqrt{\xi \pi x} e^{-x/(4 \xi)} \quad (x \to \infty). \]

6. In many applications, for example in quantum mechanics, we are interested in solutions near an irregular singular point \( x_0 \) (usually \( x_0 = 0 \) or \( x_0 = \infty \)) of the ODE
\[ y''(x) = Q(x)y(x), \quad x \geq 0. \]
In the case \( x_0 = 0 \), with \( Q(x) > 0 \) for simplicity (at least for \( x > 0 \) with \( x \) sufficiently close to 0), use the method of dominant balance to deduce that
\[ y(x) \sim \text{const.} [Q(x)]^{-1/4} \exp \left\{ \pm \int Q(x)^{1/2} \, dx \right\}, \quad (x \to 0_+). \]
Something special happens if \( Q(x) = x^{-4} \). Can you see what it is?

7. Consider Stieltjes’ ODE
\[ x^2 y''(x) + (1 + 3x)y'(x) + y(x) = 0. \quad (†) \]
Show that \( x = 0 \) is an irregular singular point. Show also that an attempt to find a solution in Frobenius’ form \( y(x) = x^\alpha \sum_{n=0}^{\infty} a_n x^n \) leads to the series
\[ a_0(1 - x + 2!x^2 - 3!x^3 + \ldots). \quad (‡) \]
What is the radius of convergence of \((‡)\)? This series gives the full A.E. of one solution of \((†)\) as \( x \to 0_+ \) (cf. Sheet 1, Qu. 6, and lecture notes). Use the method of dominant balance to deduce that another solution behaves like \( e^{1/x} / x \) as \( x \to 0_+ \). Then check that \( e^{1/x} / x \) is in fact an exact solution of \((†)\).