1* Which of the following are correct as $x \to \infty$? For any that are incorrect, provide a correct statement if you can.

\[
(x \log x)^2 = o(x^3) \quad \text{cosec} \left( \frac{x}{a^2 + x^2} \right) = O(x)
\]

\[
\tanh x = 1 + O(e^{-x}) \quad \log \left( \frac{x^2 + x}{2} \right) \sim 2 \log x
\]

\[
\frac{1}{x^2} = o(-8.2) \quad \sin \frac{x}{x} \sim 0
\]

2. If $0 < \delta < \frac{\pi}{2}$ is a constant, show that $\cosh z \sim \frac{1}{2} e^z$ as $z \to \infty$ along any path in the sector $|\arg(z)| \leq \frac{1}{2} \pi - \delta$, but not so in the sector $|\arg(z)| < \frac{1}{2} \pi$.

*Hint:* With $z = re^{i\theta}$, we have $|e^{-2z}| = e^{-2r \cos \theta}$; $r$ is approaching $\infty$, but what is happening to $\cos \theta$?

3. Which of the following are asymptotic sequences?

\[
\left\{ x^n e^{-nx} \right\}_{n=0}^{\infty} \quad (x \to \infty) \quad \left\{ \frac{1}{x^n \log x} \right\}_{n=0}^{\infty} \quad (x \to \infty) \quad \left\{ e^{(n-n^2)/x} \sin(nx) \right\}_{n=1}^{\infty} \quad (x \to 0).
\]

4.* Show that

\[
(1 + x^{-1}) \sin(x + x^{-1}) = w_1(x) \sin x + w_2(x) \cos x
\]

where $w_1(x) \sim 1$ and $w_2(x) = o(1)$ as $x \to \infty$, but that it is not valid to write

\[
(1 + x^{-1}) \sin(x + x^{-1}) \sim \sin x \quad (x \to \infty).
\]

5.* Consider

\[
I(x) = \int_{0}^{x} e^{t^2} \, dt, \quad 0 < x < \infty.
\]

Write

\[
\int_{0}^{x} \ldots = \int_{0}^{a} \ldots + \int_{a}^{x} \ldots
\]
where $a$ is an arbitrary positive number, and integrate by parts in the second of these. Considering the size of the remaining two integrals on the RHS, deduce that

$$I(x) \sim \frac{e^{x^2}}{2x} \quad (x \to \infty)$$

**Hint:** Use l’Hopital’s Rule to deduce that

$$\frac{1}{2} \int_a^x \frac{e^{t^2}}{t^2} \, dt = o\left(\frac{e^{x^2}}{2x}\right) \quad (x \to \infty).$$

6.* Show carefully that

$$y(x) = \int_0^\infty \frac{e^{-t}}{(1 + xt)} \, dt, \quad x \geq 0$$

satisfies $y(0) = 1$ and $y(\infty) = 0$, and also

$$x^2y''(x) + (1 + 3x)y'(x) + y(x) = 0.$$  

7 Find the asymptotic expansion as $x \to \infty$ of the Fourier Integral

$$I(x) = \int_{-\pi}^\pi \cos(xt) \log(t + 2\pi) \, dt.$$  

8. Use Laplace’s method to deduce that

$$\int_0^\infty e^{-x \sinh^2 t} \, dt \sim \frac{1}{2} \sqrt{\frac{\pi}{x}} \quad (x \to \infty).$$

9.* Use Watson’s lemma to deduce that

$$\int_0^5 \frac{e^{-xt}}{1 + t^2} \, dt \sim \frac{0!}{x} - \frac{2!}{x^3} + \frac{4!}{x^5} - \frac{6!}{x^7} + \ldots \quad (x \to \infty)$$