Ex: The modified Bessel function $K_\nu(x)$ has the integral representation (cf. Abramowitz & Stegun)

$$K_\nu(x) = \int_1^\infty (s^2-1)^{-\nu/2} e^{-sx} \, ds$$

[Check that this satisfies]

$$x K_\nu'(x) + K_\nu'(x) - x K_\nu(x) = 0$$

$$\left( 0 < x < \infty \right)$$

$$K_\nu(x) \to 0 \quad x \to \infty$$

To apply Watson's Lemma, put $s = t+1$ so that

$$K_\nu(x) = e^{-x} \int_1^\infty \frac{e^{-xt}}{\sqrt{t^2+x}} \, dt$$

Here

$$(e^t + xt)^{-\nu/2} = (2t)^{-\nu/2} \left( 1 + \frac{xt}{2t} \right)^{-\nu/2}$$

$$= (2t)^{-\nu} \sum_{n=0}^{\infty} \left( -\frac{xt}{2t} \right)^n \frac{\Gamma(n+1/2)}{n! \Gamma(1/2)} \quad (|t| < 1)$$

(binomial theorem).
In this case, \( \alpha = -\frac{1}{2}, \beta = 1 \) and Watson's Lemma gives

\[
K_0(x) \sim e^{-x} \sum_{n=0}^{\infty} \frac{(-1)^n \left[ \Gamma(n + 1/2) \right]^2}{n! \Gamma(1/2) x^{n + 1/2}}
\]

i.e.

\[
K_0(x) \sim \sqrt{\frac{x}{2\pi}} e^{-x} \left(1 - \frac{1}{8x} + \frac{(xy)}{2! (8x)^2} - \frac{(xy)(xy)}{3! (8x)^3} + \ldots \right)
\]

Going back to

\[
I(x) = \int_0^b e^{-x \cosh \theta} \, d\theta
\]

\[
= \int_0^a e^{-x \cosh \theta} \, d\theta + \int_a^b e^{-x \cosh \theta} \, d\theta
\]

we get our previous result rigorously now by

converting to a form where Watson's Lemma applies:

Put \( \cosh \theta = 1 + t, \sinh \theta \, dt = dt, \, dt = \frac{dt}{\sqrt{t^2 + 1}} \)

\[
\int_0^a e^{-x \cosh \theta} \, d\theta = e^{-x} \left[ \int_0^{\cosh a - 1} e^{-xt} \, dt \right] + \int_a^b e^{-x \cosh \theta} \, d\theta
\]

\[
\int_0^a e^{-x \cosh \theta} \, d\theta = \frac{e^{-x}}{\sqrt{x}} \left[ \int_0^{\cosh a - 1} e^{-xt \sqrt{x}} \, dt \right]
\]
Here
\[ f(t) = t^{-\frac{1}{2}} (1 - \frac{t}{x} + \ldots) \quad 0 < t < 2 \]
so
\[ \alpha = -\frac{1}{2}, \quad \beta = 1, \quad a_0 = 1, \quad a_1 = -\frac{x}{2}, \ldots \]

See each of the two integrals gives the same A.E.
as \[ K_0(x) \]
so \[ I(x) \sim \sqrt{\frac{2x}{\pi}} e^{-x} \left[ \frac{\Gamma\left(\frac{1}{2}\right)}{x^{1/2}} - \frac{1}{2} \frac{\Gamma\left(\frac{3}{2}\right)}{x^{3/2}} + \ldots \right] \]
\[ \text{i.e.} \quad I(x) \sim \sqrt{\frac{2x}{\pi}} e^{-x} \left[ 1 - \frac{1}{x} + \ldots \right] \sim 2K_0(x) \quad (x \to \infty) \]
But note that
\[ I(x) \neq 2K_0(x) \]

More generally, consider a \textit{Laplace integral}
\[ I(x) = \int_{a}^{b} f(t) e^{x \gamma(t)} \, dt \]
where \( f, \gamma \) are conts., \( a < b \) and \( a, b \)
are finite (or one is infinite).
General idea: If \( \varphi \) has its max. on \([a, b]\) at \( t = c \) and if \( f(c) \neq 0 \) then the nbhd of \( t = c \) gives the dominant behaviour for large \( x \).

If \( \varphi \) has several maxima, break \([a, b]\) into several subintervals, on each of which we have monotonic behaviour:

Usually only a small number of such contributions will be significant, but not always so.

In any event, we limit ourselves to consideration of

Case (A): \( \varphi \) monotonic incr. on \((a, b)\)
\[ \dot{\varphi}(b) > 0, \quad \dot{\varphi}(b) > 0 \quad \text{on} \quad (a, b) \]

Case (B): \( \varphi \) monotonic decr. On \((a, b)\)
\[ \dot{\varphi}(a) < 0, \quad \dot{\varphi}(b) < 0 \quad \text{on} \quad (a, b) \]
Then make change of variable so that integral takes the form

\[ I(x) = \int_0^T e^{-xt} f(t) \, dt \]

and apply Watson’s Lemma (assuming conditions apply!)

**Ex:** $I(x)$ as before.

More generally, in Case (A) with $\dot{\phi}(b) > 0$ can see form we will get for the leading term as follows:

\[ e^{-x\phi(b)} I(x) = \int_a^b f(t) e^{-x(\phi(b) - \phi(t))} \, dt \]

Put $s = \phi(b) - \phi(t)$, $ds = -\phi'(t) \, dt$, $dt = -\frac{ds}{\phi'(t(s))}$

\[ e^{-x\phi(b)} I(x) = \int_a^b \frac{f(t(s))}{\phi'(t(s))} e^{-xs} \, ds \]

If $f(b) \neq 0$ then apply Watson’s Lemma:

\[ \frac{f(t(s))}{\dot{\phi}(t(s))} = \frac{f(b)}{\phi'(b)} + O(s) \quad (s \to 0^+) \]

\[ \alpha = 0, \beta = 1, \quad a_0 = \frac{f(b)}{\phi'(b)} \]
So \[ e^{-x} \Psi(b) \cdot I(x) \sim \frac{f(b)}{\Psi(b)} \quad (x \to \infty) \]

\[ I(x) \sim \frac{f(b)}{\Psi(b)} \cdot e^{\Psi(b)x} \quad (x \to \infty). \]

Similarly, for Case (B) with \( \dot{\Psi}(a) > 0 \) we get

\[ I(x) \sim -\frac{f(a)}{\dot{\Psi}(a)} \cdot e^{\dot{\Psi}(a)x} \quad (x \to \infty). \]

No point in trying to give general formula for higher terms here - gets complicated - work out in any particular case using Watson's Lemma.

In the case that \( \dot{\Psi}(t) \) vanishes at end point, we can still try and convert to a form where Watson's Lemma applies, or proceed more directly as follows to see form we get for leading term:

Consider Case (B), assuming \( \dot{\Psi}(a) = 0, \quad \ddot{\Psi}(a) < 0 \)
Then

\[ I(x) = \int_a^b f(t) e^{x\varphi(t)} dt \]

\[ \sim f(a) e^{x\varphi(a)} \left[ \frac{-\pi}{2x\varphi(a)} \right]^{1/2} \quad (x \to \infty) \]

For a proof, see Copson *Asymptotic Expansions* (QA312.C58 1965)

See as follows (cf. example we started with):

\[ I(x) \sim \int_a^{a+\varepsilon} f(t) e^{x\varphi(t) + \frac{1}{2}(t-a)^2 \varphi''(a)} dt \quad (x \to \infty) \]

\[ \sim f(a) e^{x\varphi(a)} \int_a^{a+\varepsilon} e^{\frac{1}{2}(t-a)^2 \varphi''(a)} dt \quad (x \to \infty) \]

Put \( u = (t-a)\sqrt{-\varphi''(a)x} \)

\[ \sim f(a) \sqrt{-\varphi''(a)x} e^{x\varphi(a)} \int_0^\varepsilon e^{-u^2} du \]

\[ \sim f(a) \sqrt{-\varphi''(a)x} e^{x\varphi(a)} \int_0^\infty e^{-u^2} du \]

\[ \sim f(a) e^{x\varphi(a)} \left[ \frac{-\pi}{2x\varphi(a)} \right]^{1/2} \]

See for

We will get

\[ I(x) \sim f(c) e^{x\varphi(c)} \left[ \frac{-2\pi}{x\varphi(c)} \right]^{1/2} \quad (x \to \infty) \]
\[ \Gamma(x+1) = \int_0^\infty e^{-t} t^x \, dt = \int_0^\infty e^{-t} e^{x \log t} \, dt \]

Here could take \( f(t) = e^{-t}, \quad \varphi(t) = \log t \) - but not then in appropriate form as max. value of \( \varphi(t) \) is \( \infty \) at \( t = \infty \).

So, change variable: \( t = x \tau, \quad dt = x \, d\tau \)

\[ \Gamma(x+1) = \int_0^\infty e^{-x \tau} e^{x (\log x + \log \tau)} x \, d\tau = x^{x+1} \int_0^\infty e^{x (-\tau + \log \tau)} \, d\tau \quad \text{(X)} \]

Now
\[ f(t) = 1 \]
\[ \varphi(t) = -\tau + \log \tau \]
\[ \dot{\varphi}(t) = -1 + \frac{1}{\tau} \]

and \( \varphi(t) \) has peak at \( t = 1 \) with \( \dot{\varphi}(1) = -1 \)

Then
\[ x(-\tau + \log \tau) = x [\,-1 - t (t-1)^x + \cdots \,] \quad (x \to 1) \]
\[ \Gamma(x+1) \sim x^{x+1} e^{-x} \int_{1-e}^{1+e} e^{-\frac{1}{2} x (z-1)^2} \, dz \quad (x \to \infty) \]

\[ \sim x^{x+1} e^{-x} \int_{-\infty}^{\infty} e^{-\frac{1}{2} x (z-1)^2} \, dz \quad (x \to \infty) \]

i.e.

\[ \Gamma(x+1) \sim \sqrt{2\pi x} \; x^x \; e^{-x} \quad \text{Stirling's approximation!} \quad (x \to \infty) \]

To get higher order terms, put \(-z + \log z = -1 - \frac{5}{6} \log x\)

\[ \rightarrow z = 1 + \sqrt{5} \pi + \frac{1}{3} \pi^2 + \frac{1}{9\sqrt{5}} \pi^3 - \ldots \quad (x \to \infty) \]

\[ \rightarrow \Gamma(x+1) \sim \sqrt{2\pi x} \; e^{-x} \; x^x \left( 1 + \frac{1}{12} x + \frac{1}{288} x^2 + \ldots \right) \quad (x \to \infty) \]

[How spot change of variable? If consider \(e^{-t} e^{tx \log t}\)

see is max. at \(t = x\) - movable. Suggests \(t = x^2\)]
Method of Stationary Phase

Recall that we considered Fourier integrals

\[ I(x) = \int_a^b f(t) e^{ixt} \, dt \]

where \( f \) is \( N \) times continuously differentiable, and by repeated integration by parts and use of R-L Lemma, we deduced that the asymptotic behaviour of the integral as \( x \to \infty \) is governed by the behaviour of \( f \) and its derivatives at the end points \( t = a, b \) viz.

\[ I(x) \sim \sum_{n=0}^{\infty} \frac{e^{i\pi n}}{x^{n+1}} \left\{ e^{i\pi x f'(b)} - e^{i\pi x f'(a)} \right\} \quad (x \to \infty) \]

We now consider generalized Fourier integrals:

\[ I(x) = \int_a^b f(t) e^{ix\mathcal{U}(t)} \, dt \]

\( f, \mathcal{U}, a, b, x \) real.
The R-L Lemma can be extended to cover such integrals: says \( I(x) \to 0 \) as \( x \to \infty \) provided \( f \) absolutely integrable, \( \psi \neq \psi \) conts. on \([a, b]\), and \( \psi \) not const. on any subinterval of \([a, b]\).

Suppose we try integration by parts:

\[
I(x) = \left. \frac{e^{ix} \psi(t)}{ix \psi(t)} \right|_{t=a}^{t=b} - \frac{1}{ix} \int_{a}^{b} \frac{d}{dt} \left[ \frac{f(t)}{\psi(t)} \right] e^{ix} \psi(t) \, dt
\]

This gives leading asymptotic behaviour provided \( f(t)/\psi(t) \) is finite at \( t=a, b \) and nonvanishing at \( t = a, b \) and its derivative is absolutely integrable on \([a, b]\). For then generalized R-L Lemma applies to RH integral, and

\[
I(x) \sim \left[ \frac{e^{ix} \psi(b)}{ix \psi(b)} - \frac{e^{ix} \psi(a)}{ix \psi(a)} \right] \frac{1}{x} + o(\frac{1}{x})
\]

\((x \to \infty)\)
This is direct generalization of what we did in ordinary Fourier integral case, which is recovered here when \( \gamma(t) = t \).

This procedure may break down if \( \gamma(t) = 0 \) for some point or points \( t \in [a, b] \). We call these stationary points of \( \gamma(t) \). When these are present, \( I(x) \to 0 \) as \( x \to \infty \), but typically less rapidly than \( 1/x \).

Some history: G. Stokes (19) was interested in the function

\[
I(x) = \int_0^\infty \cos [x (\omega^3 - \omega)] \, d\omega \quad (x \to \infty)
\]

Phase: \( \omega = x (\omega^3 - \omega) \)

Note that this of the type we have described:

\[
f = 1, \quad \gamma(t) = t^3 - t
\]
Stokes noted that for large $x$, $\cos \Theta$ is rapidly oscillating function of $\Theta$ except near point $\Theta_0 = \frac{1}{\sqrt{x}}$ where $\frac{d\Theta}{dx} = 0$. As we move along $\Omega$-axis, $\cos \Theta$ alternates rapidly in sign; except in immediate nbhd of $\Theta = \Theta_0$ positive and negative loops of $\cos \Theta$ become more and more closely packed as $x$ grows. See Figs. 9.

Stokes argued that the contributions of these loops to the integral should tend to cancel out as $x \to \infty$ and major contribution to integral should come from immediate nbhd of $\Theta = \Theta_0$.

Because

$$\Theta^3 - \Theta^2 (\Theta^2 - 1) + 0 + \frac{1}{2!} (6 \Theta_0) (\Theta - \Theta_0)^2 + \cdots$$

$$(\Theta - \Theta_0)$$
Fig. 9a

\[ y = \cos(50(\omega^2 - \mu)) \]

Fig. 9b

\[ y = \cos[50(\omega^2 - \mu)] \]
we have
\[ I(x) \sim \int_0^\infty \cos [x (\omega_0^2 - \omega^2) + 3x \omega_0 (\omega_0 - \omega)^2] \, d\omega \]
\[ \sim \int_0^\infty \cos [x (\omega_0^2 - \omega^2) + 3x \omega_0 (\omega_0 - \omega)^2] \, d\omega \quad (x \to \infty) \]
i.e.
\[ I(x) \sim \left( \frac{\pi}{3x \omega_0} \right)^{1/4} \cos [x (\omega_0^2 - \omega^2) + \frac{\pi}{4}] \quad (x \to \infty) \]
using
\[ \int_0^\infty \cos (A + Bu^2) \, du = \sqrt{\frac{\pi}{B}} \cos (A + \frac{\pi}{4}) \quad (\text{check!}) \]
The method of stationary phase gives the leading asymptotic behaviour of generalized Fourier integrals having stationary points. It is similar in character to Laplace's method, but the reasons for success are quite different (cancellations vs. vanishingly small contributions).
Summary:

- Know how to use Watson's Lemma, in particular for Laplace integrals

\[ I(x) = \int_a^b f(t) e^{x\rho(t)} dt \]

- Two sorts: ones where \( \rho(t) \) does not vanish, and ones where it does vanish at the max. of \( \rho(t) \) on \( [a, b] \).

- Important example: Stirling's Approximation.

- Understand basic cancellation idea of Method of Stationary Phase.