Asymptotic sequences and series

Consider a sequence of functions \( \{ \varphi_n(z) \}_{n=0}^{\infty} \)

all defined for all \( z \) in some sector

\[ R = (r, \infty) e^{i(\alpha, \beta)} \]

and all nonzero for \( z \in R \) (or at least, for suff. large \( |z| \) )

Such a sequence is called an asymptotic sequence if

\[ \varphi_{n+1} = o(\varphi_n) \quad (z \to \infty), \]

\[ n = 0, 1, 2, \ldots \]

i.e.

\[ \lim_{z \to \infty} \frac{\varphi_{n+1}(z)}{\varphi_n(z)} = 0, \quad n = 0, 1, 2, \ldots \]

Exs:

1) \[ \varphi_n(z) = \frac{1}{z^n} \quad (z \to \infty) \]
2) \( f_n(z) = \frac{\log z (\sin z)^n}{(z+1)^n} \quad (z \to \infty) \)

[Here sectors are arbitrary - but watch for branch cut of \( \log \)]

Real Exs:

1) \( f_n(x) = \frac{1}{(x^2+1)^n} \quad (x \to \infty) \)

2) \( \frac{1}{x}, \frac{1}{x \ln x}, \frac{1}{x^2}, \frac{1}{x^2 \ln x}, \ldots \)

Next suppose that we are given \( f(x) \) defined on \( \mathbb{R} \) and that there exist constants \( c_0, c_1, c_2, \ldots \) such that
0) \[ f(z) = O\left( y_0(z) \right) \quad (z \to \infty) \]

1) \[ f(z) = c_0 y_0(z) + O\left( y_1(z) \right) \quad (z \to \infty) \]

2) \[ f(z) = c_0 y_0(z) + c_1 y_1(z) + O\left( y_2(z) \right) \quad (z \to \infty) \]

\[ \vdots \]

N+1) \[ f(z) = \sum_{n=0}^{N} c_n y_n(z) + O\left( y_{N+1}(z) \right) \quad (z \to \infty) \]

Note A: Second formula implies the first, and so improves on (is a refinement of, is stronger than) the first, i.e.

\[ c_0 y_0(z) + O\left( y_1(z) \right) = (c_0 + O(1)) y_0(z) = O\left( y_0(z) \right) \quad (z \to \infty) \]

so \( 1) \implies 2) \)

Similarly

\[ \vdots \implies (N) \implies (N-1) \implies (N-2) \implies \cdots \implies (0) \]
Note B: We can write, equivalently

\[ f(x) = \sum_{n=0}^{N} c_n \varphi_n(x) + o(\varphi_n(x)), \quad N = 0, 1, 2, \ldots \quad (x \rightarrow \infty) \]

(Why? Make sure you can see it!)

Note C:

\[ f \sim c_0 \varphi \quad (x \rightarrow \infty) \]

and

\[ f \sim c_0 \varphi + c_1 \varphi \quad (x \rightarrow \infty) \]

and

\[ \vdots \]

and

\[ f \sim \sum_{n=0}^{N} c_n \varphi_n \quad (x \rightarrow \infty) \quad N = 0, 1, 2, \ldots \]

We write

\[ f(x) \sim \sum_{n=0}^{\infty} c_n \varphi_n(x) \quad (x \rightarrow \infty). \]
The RHS here is called an

*asymptotic series (A.S.)* for \( f(z) \) \( (z \to \infty) \)

or an

*asymptotic expansion (A.E.)* of \( f(z) \) \( (\text{near } z = \infty) \)

The names are due to Poincaré. The first

(nonzero) term on the RHS is called the

*dominant or leading term.*

Ex: \( f(z) = e^{1/z} \) \( (z \to \infty) \)

\[ f(z) = O(1) \quad (z \to \infty) \]

\[ f(z) = 1 + O\left(\frac{1}{z}\right) \quad (z \to \infty) \]

\[ f(z) = 1 + \frac{1}{z} + O\left(\frac{1}{z^2}\right) \quad (z \to \infty) \]

\[ f(z) = 1 + \frac{1}{z} + \frac{1}{2!z^2} + O\left(\frac{1}{z^3}\right) \quad (z \to \infty) \]

\[ f(z) \sim \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{z^n} \quad (z \to \infty) \]
In this case \( \sim \) can be replaced by \( = \) everywhere \( a \neq 0 \), not just as \( z \to \infty \).

The correctness of each line follows from the convergence of the P.S. for \( e^z \) (everywhere):

\[
\sum_{n=0}^{N} \frac{z^n}{n!} = O(z^{N+1}) \quad \text{as} \quad \sum_{n=0}^{N} \frac{1}{n! z^n} = O(z^{-N-1})
\]

(\( z \to 0 \)) \quad (\( z \to \infty \))

However, this example is not especially interesting. More interesting to note that an A.S. is not always convergent, and may not converge for any \( a \).

**Ex:** Consider the function \( G(z) \) defined for \( a \) with positive real part (i.e. on \( R = (0, \infty) \)) by

\[
G(z) = \int_{z}^{\infty} \frac{e^{-as}}{1 + t} \, dt
\]
Now integrate by parts (this is OK here -- consider $\int_0^b$ and let $b \to \infty$ later):

$$
G(t) = \left[-\frac{1}{2} \frac{e^{-2t}}{1+t}\right]_{t=0}^{t=\infty} + \frac{1}{2} \int_0^\infty e^{-2t} \frac{d}{dt} \left[\frac{1}{1+t}\right] dt
$$

$$
= \frac{1}{2} - \frac{1}{2} \int_0^\infty \frac{e^{-2t}}{(1+t)^2} dt
$$

$$
= \frac{1}{2} - o\left(\frac{1}{2}\right) \quad (t \to \infty) \quad \text{as} \quad \int_0^\infty \frac{e^{-2t}}{(1+t)^2} dt \to 0
$$

$$
\lim_{t \to \infty} \frac{1}{2} = \frac{1}{2}, \quad t \in \mathbb{R}
$$

Integrate by parts again:

$$
G(t) = \frac{1}{2} + \left[\frac{1}{2} \frac{e^{-2t}}{(1+t)^2}\right]_{t=0}^{t=\infty} + \frac{1}{2} \int_0^\infty \frac{e^{-2t}}{(1+t)^2} dt
$$

$$
= \frac{1}{2} - \frac{1}{2} + o\left(\frac{1}{2}\right)
$$
Repeat:

\[ g(z) = \frac{z}{2} - \frac{z^3}{2^2} + \frac{z^5}{2^3} - \cdots + \frac{(-1)^{N+1} z^N}{2^{N+1}} + o\left(\frac{1}{2^{N+1}}\right) \]

\[ (z \to \infty) \quad (z \in \mathbb{R}) \]

So

\[ g(z) \sim \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{2^{n+1}} \quad (z \to \infty) \quad (z \in \mathbb{R}) \]

Note that if \( T_n \) denotes the \( n \)-th term of this series, then

\[ \frac{T_{n+1}}{T_n} = -\frac{m+1}{2} \]

and it is evident that the series diverges for any fixed value of \( z \). See Fig. 6.
Seems strange - series produces sharper and sharper approximations in sense of \( p(33) \) but diverges at each value of \( z \). What can it all mean?

If we evaluate, say, \( G(10) \) by integrating numerically, we get

\[
G(10) \approx 0.09156...
\]

compared with

\[
0.1000 = 0.0100 + 0.0200 - 0.006 = 0.0914
\]

A good approx! Error is only about \( 0.2\% \)

Even for as small a value as \( x = 5 \)

\[
G(5) \approx 0.17042...
\]

\[
0.2 - 0.04 + 0.016 - 0.0096 = 0.1664
\]

So what's happening here? Consider error
(real $z = x$ now for simplicity):

\[ G(x) - \frac{1}{x} = G(x) - g_\infty (x) \quad (say) = -\frac{1}{x} \int_0^\infty \frac{e^{-xt}}{(t+x)^2} \, dt \]

\[ G(x) - \left( \frac{1}{x} - \frac{1}{x^2} \right) = G(x) - g_\infty (x) = \frac{2}{x^3} \int_0^\infty \frac{e^{-xt}}{(t+x)^3} \, dt \]

\[ G(x) - \sum_{n=0}^{N-1} \left( \frac{x^n}{n!} \frac{(-1)^n n!}{x^n} \right) = G(x) - g_N (x) = \frac{(-1)^N n!}{x^N} \int_0^\infty \frac{e^{-xt}}{(t+x)^{N+1}} \, dt = \mathcal{E}_N (x), \]

For $x > 0$, \[ |\mathcal{E}_N (x)| < \frac{n!}{x^n} \int_0^\infty e^{-xt} \, dt = \frac{n!}{x^{n+1}} \]

See error in taking first $N$ terms in sum is less than first neglected term, in magnitude.

So, taking first 4 terms, for example

\[ |\mathcal{E}_4 (x)| < \frac{24}{x^5} = \begin{cases} 0.00024 & x = 10 \\ 0.0024 & x = 5 \end{cases} \]

Point is:

- Convergence refers to behaviour at each fixed $\bar{z}$ as $N \to \infty$
A.E. concerned with behaviour as $z \to \infty$

for each fixed $N$.

For a convergent series we need

$$\varepsilon_N (z) \to 0 \text{ as } N \to \infty, \text{ each fixed } z,$$

whereas for an A.S. we need

$$\frac{\varepsilon_N (z)}{g_N (z)} \to 0 \text{ as } z \to \infty, \text{ each fixed } N.$$

Thus, if we put

$$g_N (z) = \sum_{n=0}^{N-1} \frac{(-1)^n n!}{z^{n+1}},$$

in our example, we have

$$\lim_{z \to \infty} \frac{g_N (z)}{C (z)} = 1 \text{ for all finite } z,$$

but

$$\lim_{N \to \infty} \frac{g_N (z)}{C (z)} \text{ undefined for all finite } z.$$

Figs 6 & 7 illustrate this. See for $N > \frac{p}{2},$ $g_N (z)$

gives a worse and worse approximation to $C (z)$.

(Would stop at NS [8] for best approx. to $C (x)$)
In this example, see for any given $\xi$ there is a value of $N \ (\approx \|\xi\|)$ s.t. $N$th term is smallest in the series. Moreover, if truncate the series just before this term, we minimise the error $\epsilon_n(\xi)$ for that value of $\xi$. See again Fig. 7. This is reasonably typical behaviour - see B & O p. 122.

*In general*, the error in an A.S. for a function is of order smaller than that of last term included (is of order not exceeding that of first term excluded), but for *finite* $\xi$ (which after all is the case when we take an evaluation in practice) the error may be *much* larger than the last term, and possibly even larger than the partial sum of
the series to that point. This must always be remembered!

**Note A:** Even if series converges, the sum may not equal \( f(\xi) \)

**Ex:**

\[
f(\xi) = 1 + e^{-2\xi} \sim 1 + \frac{0}{2\xi} + \frac{0}{3\xi^2} + \ldots
\]

\((\xi \to \infty), \quad \text{Re}(\xi) > 0\)

Here asymptotic sequence is

\[
\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}
\]

We say that \( e^{-2\xi} \) is **subdominant** to that asymptotic sequence.
Summary:

- Understand notions of asymptotic sequences, series and expansions.
- Understand example of \( G(z) \)
- Understand that A.S. are not always convergent - may not converge for any \( z \)
  Or may converge for all \( z \), but to some other function.
- Understand difference between

\[
f(z) = \sum c_n \psi_n(z)
\]

\[
f(z) \sim \sum c_n \psi_n(z)
\]

- Understand idea of optimal truncation to get best approx. for given \( z \). 

  B80 p.122 

  dB \( \leq 1.5 \)