We had with $X = \frac{1}{\epsilon}$, $Y(X) = y(x)$,

$$\epsilon Y''(X) + 2\delta Y'(X) + \delta^2 e^{-Y(X)} = 0$$

Leading order:

$$\epsilon Y''(X) + 2\delta Y'(X) = 0$$

Three possible balances:

1) $\delta = o(\epsilon)$

$$Y_0(X) = \alpha X + \beta$$

BC: $Y_0(0) = 0 \Rightarrow Y_0(X) = \alpha X$

Can't match to

$$y_0(x) = \log\left(\frac{2}{2-x}\right) \text{ as } x \to 0^+, X \to \infty$$

2) $\epsilon = o(\delta)$

$$Y_0(X) = \alpha$$

BC: $Y_0(0) = 0 \Rightarrow Y_0(X) = 0$

Can't match to

$$y_0(x) = \log\left(\frac{5-x}{2}\right) \text{ as } x \to 0^+, X \to \infty.$$
3) $\delta \sim \varepsilon$

Put $\delta = \varepsilon$

$\gamma''(x) + 2\gamma'(x) = 0$

$\Rightarrow \quad \gamma(x) = \alpha + \beta e^{-2x}$

So

$\gamma(x) = \alpha (1 - e^{-2x})$

B.C.: $\gamma(0) = 0 \Rightarrow \beta = -\alpha$

Matching:

$\gamma_0(x) = \log \left( \frac{x}{3-x} \right) \quad x \to 0^+$

$\gamma_0(x) = \alpha (1 - e^{-2x}) \quad x \to \infty$

So

$\alpha = \log \frac{3}{e}$

and

$\gamma(x) = \log \frac{3}{e} (1 - e^{-2x})$. 
Then we have

\[ y_{\text{unif}}(x) = \log \left( \frac{3-x}{x} \right) + \log \frac{\xi}{2} \left( 1 - e^{-2x} \right) - \log \frac{\xi}{2} \]

\[ = \log \left( \frac{3-x}{x} \right) - \log \frac{\xi}{2} e^{-2x/\xi} \]

and \[ y(x) \approx y_{\text{unif}}(x), \xi \to \infty. \] See Fig. 22

with \[ \xi = 0.1 \]

*The JWKB Method*

(Jeffreys, Wentzel, Kramers, Brillouin – also Rayleigh)

This provides a powerful way of obtaining approx. solns to *linear* ODEs that cannot be solved in closed form. It is related to the device we used to obtain the behaviour of a soln near an irregular
Fig. 2a

Non-linear boundary layer problem:

\[ \varepsilon y''(x) + 2y'(x) + e^{-y(x)} = 0, \quad y(0) = y(1) = 0 \]

\( (\varepsilon = 0.1) \)

![Graph of the solution]

\[ y_{unif}(x) = \log(3-x) - e^{-\frac{2x}{\varepsilon}} \log(2) \]
singular point. But now we do something more
general, in the context of ODEs with a small
parameter \( \varepsilon \). The method is useful for IV and
BV problems, and also for eigenvalue problems.

If \( y(x) \) is the desired soln, the JWKB
approach is to suppose

\[
(6.1) \quad y(x) \sim e^{\frac{k}{\varepsilon} \sum_{n=0}^{\infty} s_n s_n (x)} \quad \varepsilon \to 0
\]

where \( s'(x) \) goes to zero with \( \varepsilon \).

**Notation:** Means

\[
\begin{align*}
y & \sim e^{s_0 s + s_1} \\
& \sim e^{s_0 s + s_1 + s_2} \\
& \sim e^{s_0 s + s_1 + s_2 + s_3} \\
& \ldots \\
& \sim e^{s_{n-1} s + s_n} \\
& \quad \varepsilon \to 0
\end{align*}
\]

and also (see later)

\[
\log [y(x)] \sim \frac{k}{\varepsilon} \sum_{n=0}^{\infty} s_n s_n (x) \quad \varepsilon \to 0
\]
Ex: \( \varepsilon^3 y''(x) = Q(x) y(x), \) \( Q(x) \neq 0 \)

Sub. (16.1) in (16.2) and assume OK to differentiate term by term:

\[
y'(x) \sim \left[ \frac{1}{6} \sum \delta^n s_n \right] \exp \left[ \frac{1}{6} \sum \delta^n s_n \right] \quad (\delta \to 0)
\]

\[
y''(x) \sim \left\{ \frac{1}{6^2} \left( \sum \delta^n s_n' \right)^2 + \frac{1}{6} \sum \delta^n s_n' s_n \right\} \exp \left[ \frac{1}{6} \sum \delta^n s_n \right]
\]

Then

\[
\frac{\varepsilon^4}{6^2} \left( s'_0 \right)^2 + \frac{2 \varepsilon^4}{6} s'_0 s' + \frac{\varepsilon^4}{6} s'' + \ldots \sim Q(x)
\]

\( \varepsilon, \delta \to 0 \)

Largest order term on LHS is \( \frac{\varepsilon^4}{6^2} \left( s'_0 \right)^2 \) by dominant balance must have the same order as \( Q(x) \).
Therefore $\delta \ll \varepsilon$ and for simplicity take $\delta = \varepsilon$.

Then

\[ (S_0')^2 + \varepsilon \left[ 2(S_0'S_0') + S_0'' \right] + \ldots \sim O(\varepsilon^2), \quad \varepsilon \to 0. \]

So we get a sequence of equns:

\begin{align*}
(6.3) \quad & (S_0')^2 = Q \quad \text{eikonal equation} \\
(6.4) \quad & 2 S_0'S_0' + S_0'' = 0 \quad \text{transport equation} \\
(6.5) \quad & 2 S_0'S_0' + S_{n-1}'' + \sum_{j=1}^{n-1} S_j'S_{n-j} = 0, \quad n \geq 2
\end{align*}

The soln of (6.3) is

\[ S_0(x) = \pm \int_0^x \sqrt{Q(t)} \, dt \]
Then
\[ S_0'(x) = \pm \sqrt{q(x)}, \quad S_0''(x) = \pm \frac{q'(x)}{q(x)} \]

(16.4) \Rightarrow \quad \pm 2\sqrt{q(x)} \ S_1'(x) = \pm \frac{q'(x)}{\sqrt{q(x)}} = 0

\Rightarrow \quad S_1'(x) = -\frac{1}{4} \frac{q'(x)}{q(x)}

\Rightarrow \quad S_1(x) = -\frac{1}{4} \log [q(x)] + \text{const.}

Stopping at this order, we get a pair of approx. solns to the ODE.

Approx. to general soln is then a linear combination:

\[ y(x) \sim c_1 q(x)^{-1/4} \exp \left\{ \frac{1}{\xi} \int_{\epsilon}^{x} \sqrt{q(t)} \ dt \right\} \]

\[ + c_2 q(x)^{-1/4} \exp \left\{ -\frac{1}{\xi} \int_{0}^{x} \sqrt{q(t)} \ dt \right\} \]

\[ (\xi \to 0+) \]
This is the leading order JWKB approxn to soln of the ODE (6.8). It can be shown that the ratio of this to the exact soln differs from 1 by terms $O(\varepsilon)$ as $\varepsilon \to 0_+$. (wherever $Q(x) \neq 0$).

Here $c_1, c_2$ are arbitrary consts. to be determined by ICs and BCs, and $Q$ is an arbitrary fixed const.

**Ex: An IV problem.**

\[ y(0) = A, \quad y'(0) = B \]

Choose $Q = 0$.

Then

\[ (c_1 + c_2) [Q(0)]^{-1/4} = A \]

\[ -\frac{1}{4} \frac{d}{d(c_1 + c_2)} [Q(0)]^{-5/4} (c_1 + c_2) + (c_1 - c_2) \frac{[Q(0)]^{1/4}}{2} = B \]
Solve for \( c_1, c_2 \quad \text{E.g.} \)

\[
A = 0, \quad B = 1 \Rightarrow \\
\begin{align*}
    c_1 &= -\frac{1}{3} \varepsilon \\
    c_2 &= \frac{1}{3} \varepsilon \left[ \frac{1}{2} \int_0^x \sqrt{Q(t)} \, dt \right]^{-1/4}
\end{align*}
\]

\( y(x) \approx \varepsilon \left[ \frac{1}{2} \int_0^x \sqrt{Q(t)} \, dt \right]^{-1/4} \sinh \left[ \frac{1}{2} \int_0^x \sqrt{Q(t)} \, dt \right] \)

Then if, for example,

\( Q(x) = \left[ 2 + \cos(x) \right]^4 \)

\( \sqrt{Q(x)} = 2 + 4 \cos(x) + \frac{1}{3} [\cos(2x) + 1] \)

\( = \frac{7}{3} + 4 \cos(x) + \frac{1}{3} \cos(2x) \)

\( \Rightarrow \int_0^x \sqrt{Q(t)} \, dt = \frac{7}{6} x + 4 \sin(x) + \frac{1}{6} \sin(2x) \)

and \( (Q(x))^{-1/4} = 1/3 \)

\( \Rightarrow \\
y(x) \approx \frac{\varepsilon}{3 \left[ 2 + \cos(x) \right]} \sinh \left[ \frac{\frac{7}{6} x + 4 \sin(x) + \frac{1}{6} \sin(2x)}{\varepsilon} \right] \)

See Fig. 23?
Ex: A BV problem.

\[ \varepsilon^2 y''(x) + y(x) = 0 \]
\[ y(0) = 0, \quad y'(1) = \pi 1 \quad (\varepsilon \to 0+) \]

Exact soln is

\[ y(x) = \frac{\sin(x/\varepsilon)}{\sin(1/\varepsilon)}, \quad \varepsilon \neq \frac{1}{n\pi} \]

Note as \( \varepsilon \) gets small, we get more and more rapid oscillations - can't describe soln in terms of BLs. So as \( \varepsilon \to 0^+ \) we get a global breakdown of our soln.

In this case, JWKB gives, in leading order (\( Q(x) = -1 \) here)

\[ y(x) \sim c_1 \exp\left(\frac{i\pi x}{2}\right) + c_2 \exp\left(-\frac{i\pi x}{2}\right) \]
\[ = c_1' \cos\left(\frac{x}{2}\right) + c_2' \sin\left(\frac{x}{2}\right) \]
Imposing BCs gives

\[ c_1' = 0, \quad c_2' = \frac{1}{\sinh(1/c)} \]

So we get exact soln in this case.

**Ex: A Sturm-Liouville problem.**

\[ y''(x) + E Q(x) y(x) = 0, \quad Q(x) > 0 \]
\[ 0 < x < \pi, \quad y(0) = 0, \quad y(\pi) = 0. \]

It is known from general theory that

1) Such a problem has an infinite no. of nontrivial solns corresponding to (positive) eigenvalues

\[ 0 < E_1 < E_2 < E_3 \ldots, \quad E_n \to \infty \quad n \to \infty \]

and corresponding distinct, real eigenfunctions

\[ y_1(x), \quad y_2(x), \quad y_3(x), \ldots \]
2) Only one soln for each eigenvalue up to multiplication by arb. consts. \((i.e.\text{ eigenvalues are nondegenerate})\).

3) \[
\int_{0}^{\pi} y_m(x) y_n(x) Q(x) \, dx = 0, \quad m \neq n
\]
(orthogonality)

Can choose const. multiple of each \(y_n\) so that

\[
\int_{0}^{\pi} [y_n(x)]^2 Q(x) \, dx = 1, \quad n = 1, 2, \ldots
\]

Then

\[
\int_{0}^{\pi} y_m(x) y_n(x) Q(x) \, dx = \delta_{mn}
\]
(orthonormal)

While general theory guarantees all of this, the \(E_n, y_n(x)\) cannot often be found explicitly.
Summary:

• See how BL theory can be used even for nonlinear ODEs. Work through the example carefully, and think about how to get next order of approximation.

• Know key idea of JWKB method. See from example how method can be used in IV and BV problems.