Perturbation theory

• An array of iterative methods for obtaining approximate solutions to problems involving a small parameter $\varepsilon > 0$, usually.

• So powerful that sometimes is a good idea to introduce a parameter $\varepsilon$ into a problem that doesn't have a small parameter; then set $\varepsilon = 1$ at end to recover soln to original problem. May be only way to make progress!

• Basic philosophy: decompose tough problems into an infinite number of relatively easy ones.

Most useful when first few iterations reveal
key factors, and rest give small corrections.

- Often lead to *asymptotic expansions*.

Let's start with an elementary example:

**Roots of a cubic.**

Suppose want approxn to roots of

\[ x^3 - 4.001x + 0.002 = 0 \]

No small parameter. But here trick is obvious:

Consider

\[ x^3 - (4+\varepsilon)x + 2\varepsilon = 0 \]  \hspace{1cm} (13.1)

This is easier to handle - consider roots to be

*functions of* \( \varepsilon \)

\[ x(\varepsilon) = \sum_{n=0}^{\infty} a_n \varepsilon^n \quad (x \approx \sum_{n=0}^{\infty} a_n \varepsilon^n \text{ as } \varepsilon \to 0) \]

To get first term, set \( \varepsilon = 0 \) in (13.1):

\[ \Rightarrow x^3 - 4x = 0 \quad \Rightarrow x = 0, \pm 2 \]
Now proceed e.g. to get root near \( x = -2 \)

\[
\begin{align*}
X(\varepsilon) &= -2 + a_1 \varepsilon + a_2 \varepsilon^2 + O(\varepsilon^3) \quad (\varepsilon \to 0) \\
\end{align*}
\]

Sub. in (13.1) to get

\[
(-8+8) + (12a_1 - 4a_1 + 2 + 2) \varepsilon + (12a_2 - a_1 - 6a_1^2 - 4a_1) \varepsilon^2
\]

\[
(\varepsilon \to 0) \quad = O(\varepsilon^3)
\]

Now see power of the idea - because \( \varepsilon \) is variable, can equate coeffs of powers of \( \varepsilon \) separately to zero.

\[
\begin{align*}
\varepsilon' &\quad : \quad 8a_1 + 4 = 0 \\
\varepsilon^2 &\quad : \quad 8a_2 - a_1 - 6a_1^2 = 0 \\
\varepsilon &\quad : \quad a_1 = -\frac{1}{2}, \quad a_2 = \frac{1}{8} \\
\end{align*}
\]

So root is

\[
x_1 = -2 - \frac{1}{2} \varepsilon + \frac{1}{8} \varepsilon^2 + O(\varepsilon^3)
\]

Setting \( \varepsilon = 0.001 \) we get \( x_1 \) to great accuracy. Similarly for other roots.
A more interesting example:

**Approximate solution of an IV problem.**

Consider

\[ y''(x) = f(x) y(x), \quad x > 0 \]  
\( y(0) = 1, \quad y'(0) = 1 \)  
\( f(x) \) cont,  
\( \text{(Get } y(x) = a, y'(x) = b \text{ case by scaling } x \times y) \)

There is no closed-form soln except for special cases of \( f(x) \).

We introduce parameter \( \epsilon \) such that the problem is solvable when \( \epsilon = 0 \).

\( \text{E.g. consider} \)

\[ y''(x) = \epsilon f(x) y(x), \quad x > 0 \]  
\( y(0) = 1, \quad y'(0) = 1 \)

Easy to solve when \( \epsilon = 0 \):
Assume
\[ y(x) \sim \sum_{n=0}^{\infty} c^n y_n(x) \quad (c \to 0) \]
where
\[ y_0(0) = 1, \quad y_0'(0) = 1 \]
\[ y_n(0) = 0, \quad y_n'(0) = 0, \quad n \geq 1 \]
See
\[ y_0''(x) = 0 \implies y_0(x) = 1 + x \]
Then
\[ y_1''(x) = f(x) y_0 = (1+x) f(x) \]
\[ y_1(0) = 0, \quad y_1'(0) = 0 \]
\[ \implies y_1(x) = \int_0^x \int_0^t f(s)(1+s) \, ds \, dt \]
More generally
\[ y_n''(x) = f(x) y_{n-1}(x) \]
\[ y_n(0) = 0, \quad y_n'(0) = 0 \]
\[ \implies y_n(x) = \int_0^x \int_0^t f(s) y_{n-1}(s) \, ds \, dt \quad (n \geq 1) \]
So we get

\[ y(x) = 1 + x + \varepsilon \int_0^t \int_0^s \int_0^u f(s) \, du \, ds \, dt \]

\[ + \varepsilon^2 \int_0^t \int_0^s f(s) \left( \int_0^u f(u) \right) \, du \, ds \, dt \]

Now easy to see that if

\[ |f(t)| \leq K, \quad 0 \leq |t| \leq 1 \times 1 \]

then

\[ |y_{n}(x)| \leq \frac{\varepsilon^n (x^2 K)^n (1 + 1 \times 1)}{(2n)!} \]

Thus series converges for all \( x, \varepsilon \). See also if \( x, K \) small, then series is rapidly convergent even at \( \varepsilon = 1 \).

\[ \text{Ex: } f(x) = e^{-x} \]

\[ \Rightarrow y_0 = 1 + x \]

\[ y_0 + y_1 = -2 + 3x + 3e^{-x} + xe^{-x} \]
\[ y_0 + y_1 + y_2 = -t + \frac{15}{4} x + 4e^{-x} \\
+ 3xe^{-x} + e^{-2x} + \frac{3}{4} xe^{-2x} \]

See Fig. 17.

Compare with other methods: If
\[ f(x) = \sum_{n} f_n x^n, \quad |x| < R \]
can try
\[ y(x) = \sum_{n} a_n x^n, \quad a_0 = a_1 = 1 \]
and compare coeffs. Will converge for \( |x| < R \).

But previous series converges for all \( x \)
and works even if \( f(x) \) has no Taylor series expansion at all!

Examples so far are regular perturbation problems. More interesting is:
Fig 17

$y''(x) = e^{-x} y(x)$

$y(0) = 1, \; y'(0) = 1$

Perturbation series approx:

$y(x) = 1 + x + \varepsilon \int_0^x dt \int_0^t ds \frac{(t-s)}{s!} e^{-s} + \varepsilon^2 \int_0^x dt \int_0^t ds \int_0^s dv \int_0^{sv} dw \frac{(t-s)}{s!} e^{-s}$

$(\varepsilon = 1)$
**Singular perturbation theory.**

Here soln to unperturbed \( \varepsilon = 0 \) problem (if it exists) is typically very different in character from exact soln when \( \varepsilon \neq 0 \).

So lowest order approxn is very different from soln of unperturbed problem (if it exists).

Not like this in regular perturbation theory.

**Elementary example:**

\[ \varepsilon x'' + x - 1 = 0 \]

\[ \varepsilon = 0 \implies x = 1. \]

See part of soln (one of roots) ceases to exist when \( \varepsilon = 0 \).

One root near \( x = 1 \) — find as before.

How can we get other root *without* solving the
quadratic? Use \textit{dominant balance} to deduce root is 
\( O(\varepsilon) \) \((\varepsilon \to 0)\). Then put \( x = \frac{1}{\varepsilon} y \) and get regular behaviour for \( y \).

In the context of BV problems for ODEs, singular perturbations typically give rise to \textit{boundary layers}.

**Elementary example (exactly solvable).**

\[
\begin{align*}
\varepsilon y''(x) - y'(x) &= 0 \quad (?) \\
y(0) &= 0, \quad y(1) = 1
\end{align*}
\]

Singular because unperturbed problem

\[
\begin{align*}
- y'(x) &= 0, \\
y(0) &= 0, \quad y(1) = 1
\end{align*}
\]

Has \textit{no solution}.

No good looking for \( y(x) \) in the form

\[
y(x) \approx \sum_{n=0}^{\infty} \varepsilon^n y_n(x) \quad (\varepsilon \to 0)
\]
because \( y_0(x) \) does not exist (satisfying (13.2))

See there is a close similarity conceptually with the previous example.

The exact soln of (13.2) is

\[
y(x) = \frac{e^{x/\varepsilon}}{e^{1/\varepsilon} - 1}
\]

In Fig. 18 we see plots of \( y(x) \) vs. \( x \) for \( \varepsilon = 0.02, 0.04, 0.06, 0.08, 0.10 \)

See for \( \varepsilon \rightarrow 0^+ \) soln is close to zero except in \textit{boundary layer} to immediate left of \( x = 1 \). (In fact of width \( O(\varepsilon) \))

(For the problem \( \varepsilon y''(x) + y'(x) = 0 \)
\[
\begin{align*}
y(0) &= 0, \\
y(1) &= 1
\end{align*}
\]

the solution is close to 1 except in boundary
Graph of $y = \frac{e^{x/\varepsilon} - 1}{e^{1/\varepsilon} - 1}$ for $\varepsilon = 0.02, 0.04, 0.06, 0.08, 0.10$
layer to immediate right of \( x = 0 \). Check!

We want to see how to construct a perturbative approx. soln to an ODE whose \textit{highest} derivative is multiplied by \( \varepsilon \). Such a problem is singular because when \( \varepsilon \to 0 \) the \textit{character} of the ODE of the soln changes.

Before proceeding, need to talk about \underline{Asymptotic matching}.

Simple idea: break interval on which BV is posed into two or more overlapping subintervals. On each subinterval use perturbn theory to obtain approxn to soln, valid on that subinterval.

Matching involves ensuring that approxns agree on regions of overlap.
Summary:

- Understand basic **ideas** of perturbation theory by studying examples given.

- Appreciate concept of **singular** perturbation theory - soln of unperturbed problem may not exist, and when it does, has a different character to perturbed soln. Look at elementary examples given to understand idea.

- Study exactly solvable example to appreciate idea of a **boundary layer**.

- Basic concept of **matching** two partial solns — more next lecture.