New complications arise in the case of inhomogeneous ODEs. Now have to deal also with singularities of inhomogeneous terms. Consider for example a case where we are interested in behaviour at an ordinary point of an associated homogeneous equation, but at which inhomogeneous term is singular:

\[ y'(x) + xy(x) = \frac{1}{x^4} \quad (x \to 0) \]

[Note: We could use variation of parameters here as homog. equation is exactly solvable. Could get exact soln, or its asymptotic behaviour, this way.]
There are four possible dominant balances:

1) \( y' \sim -xy \quad x^{-\nu} = o(xy) \quad (x \to 0) \)
   \[
   \downarrow \\
   y \sim ae^{-x^{\frac{1}{2}}} \\
   \xRightarrow{y \sim x^{-\frac{1}{2}}} \quad x^{-\nu} \neq o(xy) \\
   X
   \]

2) \( xy \sim x^{-\sigma} \quad y' = o(x^{-\sigma}) \quad (x \to 0) \)
   \[
   \downarrow \\
   y \sim x^{-\frac{1}{2}} \Rightarrow y' = o(x^{-\frac{1}{2}}) \\
   X
   \]

3) \( y' \sim axy \sim px^{-\alpha} \quad (x \to 0) \)
   \[
   \downarrow \\
   y \sim ae^{\frac{a}{x^{\frac{1}{2}}}} \Rightarrow axy \neq px^{-\alpha} \\
   X
   \]

4) \( y' \sim x^{-\alpha} \quad xy = o(x^{-\alpha}) \quad (x \to 0) \)
   \[
   \downarrow \\
   y \sim -\frac{1}{2}x^{-\frac{1}{2}} \Rightarrow xy = o(x^{-\frac{1}{2}}) \\
   \checkmark
   \]

Then set
\[ y(x) = -\frac{1}{2}x^{-\frac{1}{2}} + C(x), \quad C(x) = o(x^{-\frac{1}{2}}) \]

etc.

\[ \Rightarrow y(x) \sim -\frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2x} + a_0 + \frac{1}{3}x - \frac{1}{3}a_0 x^2 + ... \]

\( a_0 \) arb \quad \( (a_0 e^{-tx^2} \text{ is comp. fn. here}) \)
Asymptotics and Nonlinear ODEs

Can't get exact solns to most linear second-order ODEs, let alone nonlinear ODEs! There is a very limited number of systematic approaches to nonlinear ODEs.

Solns of ODEs encountered in practice are regular at almost every pt. In nbhd of such a pt., Taylor series provides adequate description and also asymptotic behaviour.

The distinguishing features of a soln are its singularities. Important to determine nature and location of these.
Solns of nonlinear ODEs have a much richer spectrum of singular behaviours than linear ones. For the latter, the only possible singular pts are those of coefficients or forcing terms. These are called **fixed singularities** - do not move as we vary initial or boundary conditions (but may disappear or reappear).

For nonlinear ODEs, can have fixed singularities and others that move in the complex plane as we vary Ics or BCs. The latter are called **spontaneous** or **movable singularities**.

**Ex:** a) Linear ODE

\[ y''(x) + \frac{2x}{(x-1)(2x-1)} y'(x) - \frac{2}{(x-1)(2x-1)} y(x) = 0 \]
Regular sing. pts. at \( x = 1, \quad x = \frac{1}{2} \)

General soln:
\[
y(x) = \frac{a}{x - 1} + bx + c_a, b \quad \text{arb. const.}
\]

See no matter what ICs or BCs (determining \( a, b \)), soln has a pole at \( x = 1 \) or is regular.

b) Nonlinear ODE:
\[
y(x) y''(x) - 2 [y'(x)]^2 = 0
\]

General soln:
\[
y(x) = \frac{a}{x - c}
\]

See have pole at \( x = c \). Location is determined by ICs or BCs that fix \( a, c \).

Note also that \( y = b \) (arb. const.) is also a soln - not in general soln. Such special solns are a
common feature of nonlinear ODEs. Often
obtained from gen. soln. in the limit of certain
parameter values - in this case, let \( |a|, |c| \to \infty \)
with \( -\frac{a}{c} = b \) to get \( y(x) = b \).
Consider \textit{dominant balance} for a nonlinear ODE:

\[
(12.1) \quad y''(x) = [y'(x)]^2 + x \quad (x \to -\infty)
\]

This ODE is first of a set of six whose solns are
called the \textit{Painlevé transcendents}.

Painlevé (c.1900) considered all equations of form

\[
y''(x) = R(x,y)[y'(x)]^2 + S(x,y)y'(x) + T(x,y)
\]

having properties:

a) \( R, S, T \) are rational functions of \( y \)
with arbitrary dependence.

b) Solns can have various kinds of **fixed** singularities (poles, branch points, essential singularities), but cannot have any **movable** singularities except for poles. He found that there are 50 distinct types having these properties. Of these, 44 types are **solvable in terms of elementa**r transcendents (exponentials, trigonometric functions, polynomials, ...), or functions defined by linear 2nd-order ODEs (Bessel functions, Legendre functions etc.), or elliptic functions. The remaining 6 equations define the Painlevé transcendents. One is \((12.1)\). It can be shown that there are solns \(\sqrt{12.1}\) with an
infinite number of second-order (movable) poles along the positive \( x \)-axis. There are also solns with poles on negative \( x \)-axis. But there are solns without poles on negative \( x \)-axis, whose asymptotic behaviour we can get fairly easily using what is essentially dominant balance. No good putting \( y = e^s \) as exponentials won't cancel. Instead, look at (12.1) and guess that there are solns with \( [y(x)]^+ \sim -x \) and \( y''(x) = o(-x) \).

Try to find them: if so then

\[
\begin{align*}
(12.2) \quad [y(x)]^+ & \sim -x , \quad y(x) \sim \pm (-x)^{1/3} & \quad (x \to -\infty) \\
(12.3) \quad y'(x) & \sim \mp \frac{1}{2} (-x)^{-1/3} \\
(12.4) \quad y''(x) & \sim \pm \frac{1}{4} (-x)^{-2/3} = o(-x)
\end{align*}
\]
Seems OK. So now look for next order term.

Put

\[ y(x) = \pm (-x)^{1/2} + \varepsilon(x) \quad (12.5) \]

and assume

\[ \varepsilon(x) = o((-x)^{1/2}) \quad (x \to -\infty) \quad (12.6) \]

Then (12.1) gives

\[ \frac{-1}{4} (-x)^{-3/2} + \varepsilon''(x) = -x \pm 2 (-x)^{1/2} \varepsilon(x) + \varepsilon(x) + x \]

\[ \therefore \varepsilon''(x) = \frac{1}{4} (-x)^{-3/2} = \pm 2 (-x)^{1/2} \varepsilon(x) + \varepsilon(x) \quad (12.7) \]

But

\[ \varepsilon''(x) = o((-x)^{1/2} \varepsilon(x)) \quad (x \to -\infty) \]

so

\[ \varepsilon''(x) = \frac{1}{4} (-x)^{-3/2} = \pm 2 (-x)^{1/2} \varepsilon(x) \quad (12.8) \]

Now we assumed

\[ \varepsilon(x) = o((-x)^{1/2}) \quad (x \to -\infty) \]
If we assume also (plausible!)

\[ E'(x) = o \left( (-x)^{-\frac{1}{2}} \right) \quad (12.9) \]

\[ E''(x) = o \left( (-x)^{-3/2} \right) \quad (12.10) \]

then (12.8) gives

\[ \mp \frac{1}{4} (-x)^{-3/2} \sim \pm 2 (-x)^{1/2} E(x) \]

i.e.

\[ E(x) \sim -\frac{1}{6} (-x)^{-2} \quad (x \to -\infty) \quad (12.11) \]

This satisfies (12.6) and gives a "particular soln" of (12.8).

To get a general soln of (12.8), put

\[ E(x) = -\frac{1}{6} (-x)^{-2} + \eta(x) \] \[ \eta(x) = -\frac{1}{6} (-x)^{1/2} \] \[ \eta(x) \sim o(-x)^{1/2} \quad (x \to -\infty) \quad (12.12) \]

(12.8) \Rightarrow

\[ \eta''(x) - 3\eta(-x)^{-4} \sim \frac{1}{4} (-x)^{-3/2} \]

\[ \sim \pm 2 (-x)^{1/2} \left[ -\frac{1}{6} (-x)^{-2} + \eta(x) \right] \]

\[ \Rightarrow \eta''(x) \sim \pm 2 (-x)^{1/2} \eta(x) + \frac{1}{3} (-x)^{-2} \quad (x \to -\infty) \quad (12.13) \]
Suppose
\[ (-x)^{-4} = o\left( (-x)^{4/5} \eta(x) \right) \quad (x \to -\infty) \] \hfill (12.14)
and check consistency:
\[ \eta''(x) \sim \pm 2 (-x)^{1/10} \eta(x) \quad \text{(linear!)} \] \hfill (12.15)
Put \[ \eta(x) = e^{S(x)} \]
\[ \eta'(x) = S'(x) e^{S(x)}, \quad \eta''(x) = S''(x) e^{S(x)} + [S'(x)]^2 e^{S(x)} \]
\[ S''(x) + [S'(x)]^2 \sim \pm 2 (-x)^{1/2} \] \hfill (12.16)
Proceeding as before, we get (check!)

For lower sign:
\[ \eta(x) \sim C_1 (-x)^{-1/10} \cos[\varphi(x)] \]
\[ + C_2 (-x)^{-1/18} \sin[\varphi(x)] \] \hfill (12.17)
\[ \varphi(x) = \frac{\sqrt{2}}{5} (-x)^{5/4}, \quad C_1, C_2 \text{ arb. const.} \]

So
\[ (A) \quad y(x) \sim -(-x)^{1/2} + C_1 (-x)^{1/10} \cos[\varphi(x)] \]
\[ -\frac{1}{8} (-x)^{-3} + C_2 (-x)^{-1/18} \sin[\varphi(x)] \quad (x \to -\infty) \] \hfill (12.18)
For upper sign:

\[
y(x) \sim + \frac{(-x)^{1/2}}{2} + d_1 (-x)^{-1/8} e^{\lambda(x)} \\
- \frac{1}{6} (-x)^{-3} + d_2 (-x)^{-7/8} e^{-\lambda(x)} \quad (x \to -\infty)
\]

Not consistent with

\[
\mathcal{E}(x) = o (-x)^{1/2} \quad (x \to -\infty)
\]

unless

\[
d_1 = 0.
\]

Then

\[
y(x) \sim (-x)^{1/2} - \frac{1}{6} (-x)^{-3} + d_2 (-x)^{-7/8} e^{-\lambda(x)} \quad (12.10)
\]

(A) is more interesting - has two arb. consts. As

vary ICs, character of soln doesn't change. But (B)

has only one arb. const. - can only get for special

ICs. It is \underline{unstable} in that sense - see Fig. 16

N.B. Our analysis doesn't tell us what ICs, say at

\[x = 0\], will lead to either of these solns,
if any of them. Use trial and error on the computer to get the pictures.

**Summary:**

- Know that inhomogeneous (forcing) terms can introduce new sing. pts. for linear ODEs.
- Read discussion of Painlevé transcendents (Not examinable!)
- Follow how “dominant balance” was used in the Painlevé example.
$y(x)$ is solution of
\[ y'' = y'^2 + x, \quad x < 0, \quad y(0) = y'(0) = 0 \]

\[ y \sim -(-x)^{\frac{1}{2}} + c_1 (-x)^{-\frac{1}{2}} \cos y(x) + c_2 (-x)^{-\frac{1}{2}} \sin y(x) \]

\[ y(x) = \frac{e^{\sqrt{2}}}{5}(-x)^{5/4} \]

[No attempt made to find $c_1, c_2$.]

\[ z = y(x) \]

\[ z = -(-x)^{\frac{1}{2}} \]
$y'' = y' + x$, $y(0) = 1$, $y'(0) = 0.2719$

$y'' = y' + x$, $y(0) = 1$, $y'(0) = 0.2320$

$y'' = y' + x$, $y(0) = 1$, $y'(0) = 0.231957$