Cover.
A collection \( \mathcal{G} = \{ G_\alpha \subset X \} \) covers a set \( S \) in \( X \) if \( S \subset \bigcup_\alpha G_\alpha \).

If each of the sets \( G_\alpha \) is open in \( X \), we say that this is an open cover for \( S \).

Compact Set.
A set \( S \subset X \) is compact if and only if for every collection \( \{ G_\alpha \} \) of open sets in \( X \) which cover \( S \), there is a finite subcollection \( \{ G_{\alpha_i} \} \), \( i = 1 \ldots n \), such that

\[
S \subset \bigcup_{i=1}^{n} G_{\alpha_i}.
\]

(Every open cover has a finite subcover.)

A compact set is bounded.
Let \( S \) be a compact set in \( (X, d) \).
Choose some \( a \in X \).
The sets \( G_n = \{ x \in X : d(x, a) < n; n \in \mathbb{N} \} \) are open in \( (X, d) \), and cover \( X \).
Therefore they cover \( S \).
Therefore there is a finite set \( \{ G_{n_1}, G_{n_2}, \ldots, G_{n_k} \} \) which covers \( S \).
Therefore \( d(x, a) < \max(n_i) \) for every \( x \in S \), and \( S \) is bounded.

A compact set is closed.
Let \( S \) be a compact set in \( (X, d) \), and let \( a \in \setminus S \).
The sets \( G_n = \{ x \in X : d(x, a) > \frac{1}{n}; n \in \mathbb{N} \} \) are open in \( (X, d) \) and cover \( X \setminus \{a\} \).
Therefore they cover \( S \).
Therefore there is a finite set \( \{ G_{n_1}, G_{n_2}, \ldots, G_{n_k} \} \) which covers \( S \).
If \( N = \max n_i \), \( d(x, a) > \frac{1}{N} \) for all \( x \in S \), and \( a \) is not an accumulation point of \( S \).
Therefore \( S \) contains all its accumulation points, and is closed.

The converse of these results does not hold in general.
If \( d \) is the discrete metric and \( X \) is an infinite set, then any infinite subset \( S \subset X \) is closed and bounded, but is not compact.
The collection \( \{ G_\alpha = \{ \alpha \} : \alpha \in S \} \) is an open cover for \( S \) but no finite subcollection covers \( S \).

However, the converse does hold in \( (\mathbb{R}, |.|) \).

The Heine-Borel Theorem. A closed and bounded set in \( \mathbb{R} \) is compact.

Theorem. An infinite subset of a compact set has an accumulation point in the set.
Proof.
Suppose that \( S \) is a compact set in \( (X, d) \), the set \( T \subset S \) has no accumulation point in \( S \).
Then for every \( x \in S \), there is a neighbourhood \( \mathcal{N}(x, \epsilon_x) \) for some \( \epsilon_x > 0 \) which contains at most one point of \( T \) (when \( x \in T \)).
The collection \( \{ \mathcal{N}(x, \epsilon_x) \} \) is an open cover for \( S \), therefore there is a finite subcover for \( S \).
Since this sub-cover also covers \( T \), and each set in the sub-cover contains at most one point of \( T \), the set \( T \) is finite.

It follows that a Cauchy sequence in a compact set converges in the set.

**Theorem.** If \( S \) is a compact set, and \( f: (X,d_X) \to (Y,d_Y) \) is continuous on \( S \), then \( f \) is uniformly continuous on \( S \).

**Proof.**

Given any \( \epsilon > 0 \), for every \( x \) in \( S \) there is a \( \delta_x > 0 \) such that

\[
d_X(y,x) < \delta_x \implies d_Y(f(y), f(x)) < \epsilon/2.
\]

For each \( x \in S \), define the set \( G_x \) by

\[
G_x = \{ y \in S ; d_X(y,x) < \delta_x/2 \}.
\]

The collection \( \{G_x\} \) is an open cover for \( S \).

Therefore, there is a finite set \( \{x_1, \ldots, x_n\} \) such that \( \{G_{x_i}\} \) is an open cover for \( S \).

Let \( \delta = \min(\delta_{x_i}/2) \).

Any \( x \in S \) is in one of the \( G_{x_i} \); i.e \( d_X(x,x_i) < \delta_{x_i}/2 < \delta_{x_i} \).

\[
d_X(y,x_i) \leq d_X(y,x) + d_X(x,x_i) < \delta + \delta_{x_i}/2 \leq \delta_{x_i}.
\]

Therefore

\[
d_Y(f(y), f(x_i)) < \epsilon/2 \quad \text{and} \quad d_Y(f(y), f(x_i)) < \epsilon/2
\]

so that \( d_Y(f(y), f(x)) < \epsilon \).

i.e. \( y, x \in S \) and \( d_X(y,x) < \delta \implies d_Y(f(y), f(x)) < \epsilon \).

Consequently, if \( \{x_n\} \) is a Cauchy sequence in a compact set \( S \), and \( f \) is continuous on \( S \) then \( \{f(x_n)\} \) is a Cauchy sequence in \( f(S) \).

**Proof.**

Since \( f \) is uniformly continuous from \( S \) to \( f(S) \), given any \( \epsilon > 0 \), there is a \( \delta > 0 \) such that

\[
d_Y(f(y), f(x)) < \epsilon \quad \forall \ x, y \in S ; \ d_X(y,x) < \delta.
\]

Given this \( \delta \), we can find \( N \in \mathbb{N} \) such that

\[
d_X(x_n, x_m) < \delta \quad \forall \ n, m > N.
\]

But then

\[
d_Y(f(x_n), f(x_m)) < \epsilon \quad \forall \ n, m > N.
\]
Theorem. If $S \subset X$ is compact, and $f : X \to Y$ is continuous on $S$ then $f(S)$ is compact.

Proof. Let $\{G_\alpha\}$ be any open cover for $f(S)$. Then $\{f^{-1}(G_\alpha)\}$ is an open cover for $S$.

But $S$ is compact, therefore there is a finite subcover $\{f^{-1}(G_{\alpha_i})\}$ for $S$, and $\{G_{\alpha_i}\}$ is now a finite subcover for $f(S)$.

Combining these results, we see that if $\{x_n\}$ is Cauchy in $S$, and $f$ is continuous on $S$, $\{f(x_n)\}$ converges in $f(S)$.

Since a compact set is closed and bounded, we also have as a consequence the following.

Corollary: The extreme value theorem. If $S$ is compact, and $f; S \to \mathbb{R}$ is continuous on $S$, then there are $x_1, x_2 \in S$ such that

$$f(x_1) \leq f(x) \leq f(x_2) \text{ for all } x \in S.$$