Separation

Suppose that we have a sequence \( \{ x_n \} \) in a topological space \((X, T)\).

In order to consider the convergence of such a sequence, we consider the corresponding convergence result for a metric space in terms of open sets:

namely: A sequence converges to a limit \( l \in X \) if every (open) neighbourhood containing \( l \) contains all but a finite number of points of the sequence.

Consequently, we define convergence in \((X, T)\) by:

A sequence converges to a limit \( l \in X \) if every open set containing \( l \) contains all but a finite number of points of the sequence.

However, this definition can have some unexpected consequences.

If \( T = \{ \emptyset, X \} \), then every sequence in \((X, T)\) converges, and every \( l \in X \) is a limit of the sequence.

Similarly, if we have a sequence in MATH 3402 which converges to an element of \( S_{11} \) (say) then every element of \( S_{11} \) is a limit of the sequence.

If we want convergent sequences to have unique limits in \( X \), then we need to impose further restrictions on the topology of \( X \).

Basicly the problem arises because there are distinct points \( x_1 \) and \( x_2 \) such that every open set containing \( x_1 \) contains \( x_2 \). Hence any sequence converging to \( x_2 \) converges to \( x_1 \) also.

(In the examples above, this works both ways, but this is not necessary.

If we consider the set \( \{ a, b \} \) with the topology \( \{ \emptyset, \{ a \}, X \} \), then every open set containing \( b \) contains \( a \), but not vice-versa.)

This means that \( x_1 \) is a limit point of the set \( \{ x_2 \} \), and this latter set is not closed.

The simplest separation axiom is therefore to require that every set consisting of a single point be closed.

A topological space satisfying this axiom is called a \( T_1 \)-space.

An equivalent formulation is to require that if \( x_1 \) and \( x_2 \) are distinct points, then there is an open set \( S_1 \) containing \( x_1 \) but not \( x_2 \).

If we have the first form of the axiom, then if \( \{ x_2 \} \) is closed then \( S_1 \in \emptyset \{ x_2 \} \) is an open set containing \( x_1 \) but not \( x_2 \).

Conversely, if we take the second form, then given \( y \in X \), for each \( x \neq y \) we can find an open set \( S_y \) which contains \( x \) but not \( y \).

But then \( \emptyset \{ y \} = \bigcup S_y \) is open, and \( \{ y \} \) is closed.

Hausdorff Spaces.

A more stringent separation axiom requires that if \( x_1 \neq x_2 \), there exist open sets \( U_1 \) and \( U_2 \) such that \( x_1 \in U_1 \), \( x_2 \in U_2 \), and \( U_1 \cap U_2 = \emptyset \). (We say that \( U_1 \) and \( U_2 \) are disjoint.)

Spaces with this property are called \( T_2 \)-spaces or Hausdorff spaces.

Spaces for which the topology is derived from a metric are Hausdorff spaces.

Given \( x_1 \neq x_2 \), \( d(x_1, x_2) = r > 0 \).

Then we can choose \( U_i = \{ d(x, x_i) < \frac{r}{2} \} \).

Every Hausdorff space is obviously a \( T_1 \)-space, but some \( T_1 \)-spaces are not Hausdorff spaces.
**Regular Spaces.**

A topological space is called **regular** if for every point $x$ and each closed set $V$ not containing $x$, there exist disjoint open sets $U_1$ and $U_2$ such that $x \in U_1$ and $V \subset U_2$.

Equivalently, for each point $x$ and each open set $U$ containing $x$, there is an open set $U_1$ containing $x$ such that $\text{Cl}(U_1) \subset U$.

A regular space need not be a $T_1$-space.

For example, the set of students in MATH 3402 together with the topology $T_3$ generated by $\{S_{11}, S_{12}, S_{21}, S_{22}\}$ is regular but not $T_1$.

Therefore it is usual to consider regular $T_1$-spaces.

Such spaces are denoted $T_3$-spaces.

Every $T_3$-space is Hausdorff, but not the converse.

A similar problem arises with the next class of spaces.

**Normal Spaces.**

A topological space is **normal** if for every pair of disjoint closed sets $V_1$ and $V_2$ there exist disjoint open sets $U_1$ and $U_2$ with $V_i \subset U_i$.

The same example as above shows that a normal space need not be a $T_1$-space.

A normal $T_1$-space is denoted a $T_4$-space.

Every $T_4$-space is $T_3$, but again the converse need not apply.

We will consider mainly Hausdorff spaces.

In a Hausdorff space, a convergent sequence has a unique limit.

Suppose that $l$ is a limit of a sequence in $X$, and that $l_1$ is any other point of $X$.

Then there are disjoint open sets $U_1$ containing $l$ and $U_2$ containing $l_1$.

Since $l$ is a limit, all but a finite number of terms in the sequence belong to $U_1$,

so that at most a finite number belong to $U_2$.

Therefore $l_1$ is not a limit for the sequence.

Any subspace of a Hausdorff space is Hausdorff.

If $Y \subset X$, and $(X, T)$ is Hausdorff, then for any two distinct points $x_1$ and $x_2$ in $Y$, (and hence in $X$), there are disjoint open sets $U_1$ and $U_2$ in $T$ containing $x_1$ and $x_2$ respectively.

But then $V_1 = U_1 \cap Y$ and $V_2 = U_2 \cap Y$ are disjoint open sets in the relative topology which contain $x_1$ and $x_2$ respectively.

**Compact sets.**

A subset $S$ of a topological space $(X, T)$ is called compact if every open covering of $S$ includes a finite subfamily which covers $S$.

Since a subset consisting of a single point is trivially compact, we see that if the space is not $T_1$, a compact set need not be closed.

If $S$ is compact as a subset of $X$ then it is compact when we consider $S$ as the whole space and use the relative topology.

In this case we refer to $S$ as a compact space.
A set $S \subset X$ is compact if and only if in every family of relatively closed subsets of $S$, the intersection of all of which is $\phi$, there is a finite subfamily with the same property.

If $\{V_\alpha\}$ is a family of relatively closed subsets with null intersection, then $\{U_\alpha = S \setminus V_\alpha\}$ is an open cover for $S$.

Therefore there is a finite subcover for $S$ if and only if there is a finite subfamily of the $\{V_\alpha\}$ with null intersection.

A closed subset of a compact space is compact.

Let $S$ be a compact space, and $C$ a closed subset of $S$.

If $\mathcal{F}$ is any collection of relatively closed subsets of $C$ whose intersection is $\phi$, then for each $F \in \mathcal{F}$, $F$ is closed in $S$.

Since $S$ is compact, there is a finite subfamily of $\mathcal{F}$ whose intersection is null, and hence $C$ is compact.

We now consider compactness as applied to Hausdorff spaces.

If $C_0$ and $C_1$ are disjoint compact sets in a Hausdorff space $X$, there exist disjoint open sets $W_0$ containing $C_1$ respectively.

Consider first the case in which $C_0$ consists of a single point $x_0$ which is not in $C_1$.

(Such a set is trivially compact)

Then for every point $y \in C_1, x_0 \neq y$, and there are disjoint open sets $U_y$ and $V_y$ containing $x_0$ and $y$ respectively.

The sets $V_y$ cover $C_1$, and hence there is a finite subfamily $\{V_1, \ldots, V_n\}$ which covers $C_1$.

Let $\{U_1, \ldots, U_n\}$ be the corresponding $U_y$.

Then $W_0 = \bigcap_{i=1}^n V_i$ is an open set containing $x_0$ disjoint from each of the $V_i$, and therefore from $W_1 = \bigcup_{i=1}^n V_i$, which is an open set containing $C_1$.

Now consider the case in which $C_0$ is an arbitrary compact set disjoint from $C_1$.

For each $x$ in $C_0$, the above construction gives disjoint open sets $W_{0x}$ containing $x$ and $W_{1x}$ containing $C_1$.

The collection $\{W_{0x}\}$ is an open cover for $C_0$.

Therefore there is a finite subcollection $\{W_{01}, \ldots, W_{0n}\}$ which covers $C_0$.

Each element of the corresponding subcollection $\{W_{1i}\}$ contains $C_1$, so that $W_1 = \bigcap_{i=1}^n W_{1i}$ is an open set containing $C_1$ and disjoint from $W_0 = \bigcup_{i=1}^n W_{0i}$ which is an open set containing $C_0$.

A compact subset of a Hausdorff space is closed.

Suppose $C$ is compact, and $x \in \text{Cl}(C)$.

If $x$ is not in $C$, then there are disjoint open sets $W_0$ containing $x$ and $W_1$ containing $C$.

Therefore $x$ is not a limit point of $C$ either, which is a contradiction.

Therefore, $x \in C$ and $C = \text{Cl}(C)$.
**Locally Compact Spaces**

A topological space is **locally compact** if every point has a neighbourhood whose closure is compact.

Any topological space \((X, T_X)\) can be embedded in another topological space \((Y, T_Y)\) having just one more point than \(X\) in such a way that \(Y\) is compact and the relative topology of \(X\) as a subset of \(Y\) is the original topology of \(X\).

This process is called the **one-point compactification** of \(X\).

Let \(y\) be any point distinct from the points of \(X\).

Let \(W\) be the class of open sets \(W\) in \(X\) such that \(X \setminus W\) is compact.

Since \(\phi\) is compact, \(X \in W\).

Let \(Y = X \cup \{y\}\).

We choose for the topology \(T_Y\) of \(Y\) the topology \(T_X\) of \(X\) together with the sets \(W \cup \{y\}\) for \(W \in W\).

Note that \(Y = X \cup \{y\}\) belongs to \(T_Y\) as required.

Now consider any open covering of \(Y\).

Since it covers \(y\), it must contain at least one set of the form \(W_0 \cup \{y\}\), where \(X \setminus W_0\) is compact in \(X\).

Furthermore, \(X \setminus W_0\) is covered by the relatively open sets \(V \cap X\) where \(V\) is in the given covering for \(Y\).

Since \(X \setminus W_0\) is compact, a finite subfamily of these sets cover \(X \setminus W_0\), and these sets together with \(W_0 \cup \{y\}\) cover \(Y\).

Therefore \(Y\) is compact.

If \(X\) is a \(T_1\)-space, so is \(Y\), but it can happen that \(Y\) is not Hausdorff even though \(X\) is.

However, if \(X\) is a locally compact Hausdorff space, \(Y\) will be a Hausdorff space.

Since \(X\) as a subset of \(Y\) inherits the original topology of \(X\), any two points in \(X\) have the Hausdorff property. Therefore it suffices to consider \(x \in X\) and the point \(y\).

Since \(X\) is locally compact, there is an open set \(U\) containing \(x\) such that \(Cl(U)\) is compact.

But then \((X \setminus Cl(U)) \cup \{y\}\) is an open set in \(Y\) containing \(y\) and disjoint from \(U\). Therefore \(Y\) is Hausdorff.

Conversely, if \(X\) and \(Y\) are both Hausdorff, \(X\) is locally compact.