INTERNAL STUDENTS ONLY

THE UNIVERSITY OF QUEENSLAND

Mid Semester Examination, 21 April, 2015

MATH3401
Complex Analysis
(Unit Courses)

Time: 45 Minutes for working
No perusal time before examination begins

CREDIT WILL BE GIVEN ONLY FOR WORK WRITTEN ON THIS EXAMINATION SCRIPT.
FULL WORKING MUST BE SHOWN.
Use the back pages if the space provided is insufficient, and/or for rough working.
Answer all questions. Questions carry the marks indicated, total marks are 100.
Check that this examination paper has 12 printed pages.
NO programmable, graphing or ASCII calculators allowed.

FAMILY NAME (PRINT): ____________________________

GIVEN NAMES (PRINT): ____________________________

STUDENT NUMBER: ________________________________

SIGNATURE: ______________________________________

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1. Calculate $\frac{d}{dz}(1 + i)^2$, explaining any restrictions you need to make for your answer to be valid. [10 marks]
2. (a) Prove that \( \sinh^{-1} z = \log \left( z + (z^2 + 1)^{1/2} \right) \). Note that here, \( w \mapsto w^{1/2} \) is double valued. \([12 \text{ marks}]\)

(b) Find all solutions \( z \in \mathbb{C} \) of \( \sinh z = 4i \) (express them in the form \( x + iy \)). \([13 \text{ marks}]\)
(Question 2 continued).
3. Let $f(z) = x^2 + y^2 - 2ixy$ (where $z = x + iy$). [25 marks]

(i) Find all points $z \in \mathbb{C}$ at which $f$ satisfies the Cauchy-Riemann equations. (Hint: the set is non-empty).

(ii) Find all points $z \in \mathbb{C}$ at which $f$ is differentiable (Hint: the set is non-empty). Make sure you justify your answer.

(iii) Show that $f$ is nowhere analytic in $\mathbb{C}$.

(iv) Explain why there is no contradiction between your answers to (ii) and (iii).
(Question 3 continued).
4. (a) Show the following limits [6 marks each]:

i) \( \lim_{z \to \infty} \frac{z^9 + 7z}{z^5 + 17z} = \infty \)

ii) \( \lim_{z \to 1} \frac{a}{z - 1} = \infty \) if \( a \neq 0 \).

(b) [13 marks] Determine the Möbius transformation (viewed as a mapping on \( \mathbb{C} \)) mapping \( \infty \) to 0, 0 to \( -i \), and 1 to \( \infty \).
(Question 4 continued).
5. [15 marks] Find all solutions of \( \sin z = \cosh 4 \) in \( \mathbb{C} \). Express your answers in the form \( x + iy \).
(Question 5 continued).
extra working space
bonus extra working space
(1) \((1+i)^2 = \exp(2 \log(1+i))\), where we need to specify a single-valued \(\text{f}^n\) to differentiate, i.e., specify a branch of \(\log\), e.g. \(\text{Log}\). Then for this branch, specifically for \(-\pi < \arg z < \pi\) (\(\epsilon > 0\)), there holds:

\[
\frac{d}{dz} [(1+i)^2] = \text{Log}(1+i) \exp (2 \text{Log}(1+i)) = \text{Log}(1+i) (1+i)^2
\]

\[
= (\ln \sqrt{2} + \frac{i \pi}{4}) (1+i)^2 \]

\[
= \left(\frac{i \ln 2 + i \pi}{4}\right) (1+i)^2
\]
(2) \( w = \sinh^{-1} z \Rightarrow z = \sinh w = \frac{e^w - e^{-w}}{2} \)

So \( 2z = e^w - e^{-w} \Rightarrow 2ze^w = e^w - 1 \)

i.e. \( e^{2w} - 2ze^w - 1 = 0 \)

\[ \Rightarrow e^w = \frac{2z + (4z^2 + 4)^{1/2}}{2} \]

\[ = z + (z^2 + 1)^{1/2} \]

So \( w = \sinh^{-1} z = \log e^w = \log (z + (z^2 + 1)^{1/2}) \)

as req'd.

b) \( \sinh z = 4i \iff z = \sinh^{-1} 4i \)

\[ = \log \left( 4i + (-16 + t)^{1/2} \right) \]

\[ = \log (4i + 5(1 + t)) \]

\[ = \log \left( 4 + 5\sqrt{1} \right) + \log \left( 4 - 5\sqrt{1} \right) \]

\[ = \{ \ln |4 + 5\sqrt{1}| + (2n + 1)\pi i, n \in \mathbb{Z} \} \cup \]

\[ \{ \ln |4 - 5\sqrt{1}| + (2n + \frac{1}{2})\pi i, n \in \mathbb{Z} \} \]
(3) \( f(z) = u(x,y) + iv(x,y) \) for \( z = x + iy \),

where \( u = x^2 + y^2 \), \( v = -2xy \)

So \( u_x = 2x \) \( u_y = 2y \)
\( v_x = -2y \) \( v_y = -2x \)

\( c^\text{R_I} \Rightarrow u_x = v_y \Rightarrow 2x = -2y \Leftrightarrow x = 0 \).

\( c^\text{R_II} \Rightarrow u_y = -v_x \Rightarrow 2y = -2y \), which is always true.
So \( c^\text{R} \) hold only on the line \( \{ x = 0 \} \), i.e., the imaginary axis.

(ii) By (i), \( f \) is only possibly differentiable
on the imaginary axis (since \( c^\text{R} \) are necessary for differentiability). Since \( u, v, u_x, v_x, u_y, v_y \) are defined
& cons on \( \mathbb{R}^2 \), by (i) \( f \) is differentiable precisely
on the \( \text{Im} \) axis.

(iii) \( f \) is not on any nbhd of any pt.
in \( C \), and hence is nowhere analytic.
(iv) analytic \( \Rightarrow \) differentiable
but not vice versa, so no contradiction.
\( \lim_{z \to \infty} f(z) = \infty \iff \lim_{z \to 0} \frac{1}{f(z^2)} = 0 \)

So for \( f(z) = \frac{z^9 + 7z}{z^5 + 17z} \), \( \text{RHS} = \)

\[
\lim_{z \to 0} \frac{1}{\frac{z^9 + 7z}{z^5 + 17z}} = \lim_{z \to 0} \frac{\frac{1}{z^9} + \frac{7}{z}}{\frac{1}{z^5} + \frac{17}{z}}
\]

\[
= \lim_{z \to 0} \frac{z^4 + 17z^8}{1 + 7z^8} = 0 \text{ as reqd.}
\]

(ii) \( \lim_{z \to 1} f(z) = \infty \iff \lim_{z \to 1} \frac{1}{f(z)} = 0 \).

\( \text{RHS} = \lim_{z \to 1} \frac{1 - z}{9} = 0 \text{ as reqd.} \).
4. b) Put \( f(z) = \frac{az + b}{cz + d} \).

\[ f(\infty) = 0 \Rightarrow a = 0 \quad \text{(1)} \]

\[ f(0) = -i \Rightarrow \frac{b}{d} = -i \quad \text{(2)} \]

\[ f(1) = i \Rightarrow c = -d \quad \text{(3)} \]

So put \( d = 1 \): \( \text{(2)} \Rightarrow b = -i \), \( \text{(3)} \Rightarrow c = 1 \)

So \( f(z) = \frac{-i}{z + 1} = \frac{i}{z - 1} \).
\( 5 \) \( \sin z = \cosh 4 \):

1. for \( z = x + iy \),
\[
\sin z = \sin x \cosh y + i \cos x \sinh y
\]

2. So equating real and imaginary parts of 1 & 2:
   \[
   \sin x \cosh y = \cosh 4 \] \( \text{①} \)
   \[
   \cos x \sinh y = 0 \] \( \text{②} \)

\( \text{③} \Rightarrow x = \pm \frac{\pi}{2} + 2n\pi \quad n \in \mathbb{Z} \) \( \text{③}_+ \)

or \( y = 0 \) \( \text{③}_- \)

For \( \text{③}_+ \):
\( \text{①} \Rightarrow \cosh y = \cosh 4 \), so \( y = \pm 4 \);

For \( \text{③}_- \), \( \cosh y = 1 \), so \( \text{①} \Rightarrow \sin x = \cosh 4 > 1 \),

which is not possible.

For \( \text{③}_- \):
\( -\cosh y = \cosh 4 \), which has no sol.

Hence the only sols are
\[
\left\{ \frac{\pi}{2} + 2n\pi \pm 4i \mid n \in \mathbb{Z} \right\}
\]