MATH3401
Complex Analysis
(Unit Courses)

Time: 45 Minutes for working
No perusal time before examination begins

CREDIT WILL BE GIVEN ONLY FOR WORK WRITTEN ON THIS EXAMINATION SCRIPT.
FULL WORKING MUST BE SHOWN.
Use the back pages if the space provided is insufficient, and/or for rough working.
Answer all questions. All questions will carry equal weight (25 marks each).
Check that this examination paper has 10 printed pages.
NO programmable, graphing or ASCII calculators allowed.

FAMILY NAME (PRINT): ___________________________________________

GIVEN NAMES (PRINT): ___________________________________________

STUDENT NUMBER: ________________

SIGNATURE: ____________________________________________________

<table>
<thead>
<tr>
<th>EXAMINER'S USE ONLY</th>
</tr>
</thead>
<tbody>
<tr>
<td>QUESTION</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td></td>
</tr>
</tbody>
</table>
1. [25 marks] Find the cube roots of $-8000i$, i.e., find all solutions of $z^3 + 8000i = 0$ in $\mathbb{C}$. Express your answers in the form $x + iy$, with $x, y \in \mathbb{R}$.

$-8000i = 8000e^{-i\pi/2}$

Since $8000 = 20$, cube roots are

$20^{1/3}, 20e^{-i\pi/6}, 20e^{i(-\pi/6 + 2\pi/3)}, 20e^{i(-\pi/6 + 4\pi/3)}$

i.e. $20 \cos(-\pi/6), 20 \cos(\pi/2), 20 \cos(7\pi/6)$

i.e. $20 \left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right), 20i, 20 \left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$

i.e. $10\sqrt{3} - 10i, 20i, -10\sqrt{3} - 10i$. 
2. (a) Prove that \( \cosh^{-1} z = \log \left( z + (z^2 - 1)^{1/2} \right) \). Note that here, \( w \mapsto w^{1/2} \) is double valued. [12 marks]

(b) Find all solutions \( z \in \mathbb{C} \) of \( \cosh z = 3i \) (express them in the form \( x + iy \)). [13 marks]

\[
\text{(a)} \quad w = \cosh^{-1} z \iff z = \cosh w = \frac{e^w + e^{-w}}{2}
\]

So \( 2z = e^w + e^{-w} \Rightarrow 2ze^w = e^w + 1 \)

i.e. \( e^{2w} - 2ze^w + 1 = 0 \)

\[
\Rightarrow e^w = \frac{2z + (4z^2 - 4)^{1/2}}{2}
\]

\[
= z + (z^2 - 1)^{1/2}
\]

So \( w = \cosh^{-1} z = \log(e^w) = \log\left(z + (z^2 - 1)^{1/2}\right) \) as req'd.

\[
\text{(b)} \quad \cosh z = 3i \iff z = \cosh^{-1} 3i
\]

\[
= \log\left(3i + (-9 - 1)^{1/2}\right)
\]

\[
= \log\left(3i \pm 3i\pi\right)
\]

\[
\Rightarrow \{\ln(3 \pm 3i\pi) + (2n \pm 1)i\pi, n \in \mathbb{Z}\} \cup \{\ln(-3 \pm 3i\pi), n \in \mathbb{Z}\}
\]

Since \( \ln(-3) = -\ln(3 + 3) \), this can be rewritten as

\[
\{\ln(3 \pm 3\pi) + (2n \pm 1)i\pi, n \in \mathbb{Z}\}
\]
3. Let $f(z) = z \Re(z)$ (i.e., $z$ times the real part of $z$).
   (a) [16 marks]
   (i) Find all points $z \in \mathbb{C}$ at which $f$ satisfies the Cauchy-Riemann equations. (Hint: the set is non-empty).
   (ii) Find all points $z \in \mathbb{C}$ at which $f$ is differentiable (Hint: the set is non-empty).
   Make sure you justify your answer.
   (iii) Show that $f$ is nowhere analytic in $\mathbb{C}$.
   (iv) Explain why there is no contradiction between your answers to (ii) and (iii).

(b) [9 marks] Define the function $f(z) = \frac{z}{z}$ on $\mathbb{C} \setminus \{0\}$. Prove that $f$ cannot be continuously extended to $\mathbb{C}$, i.e., there is no choice of $f(0)$ that makes $f$ continuous on all of $\mathbb{C}$.

   a) $z = x + iy \implies f(z) = (x + iy)x = x^2 + ixy$

   i) $u = x^2, \quad v = xy$ so $u_x = 2x, \quad u_y = 0$,

   $v_x = y, \quad v_y = x$.

   CR$_u : u_x = v_y \implies 2x = x \implies x = 0$

   CR$_v : u_y = -v_x \implies 0 = y \implies y = 0$.

   So CR hold precisely at $(0,0)$.

   (ii) $u, v$ and all partials are defined and cts on $\mathbb{R}^2$, so by sufficient conditions from class (1/4), $f$ is differentiable at $0$.

   Since CR are not satisfied anywhere else, & they are necessary for differentiability, $f$ is only differentiable at $0$.

   (iii) Analytic at $z_0$ says differentiable on a nbhd of $z_0$. Since $f$ is only differentiable at one point, this means it is analytic nowhere.

   (iv) Because analytic $\iff$ differentiable, but differentiable $\neq$ analytic.
(Question 3 continued).

\[ b) \ y \neq 0 : \ f(iy) = \frac{-iy}{iy} = -1. \]
\[ \Rightarrow \lim_{z \to 0} f(z) = -1. \quad \star \]
\[ \text{on Im$\mathbb{C}$axis} \]

\[ x \neq 0 : \ f(x) = \frac{x}{x} = 1 \]
\[ \Rightarrow \lim_{z \to 0} f(z) = 1. \quad \star \star \]
\[ \text{on Re$\mathbb{C}$axis} \]

The limit must be independent of direction of approach in $f$ is $\text{cts}$ at $0$ : by $\star$ & $\star \star$, this isn't possible.
4. (a) Show the following limits [6 marks each]:

i) \[ \lim_{z \to \infty} \frac{z^7}{z^9 + 42z} = \infty \]

ii) \[ \lim_{z \to \infty} \frac{az + b}{cz + d} = \frac{a}{c} \text{ if } ad - bc \neq 0 \text{ and } c \neq 0. \]

(b) [13 marks] Determine the Möbius transformation (viewed as a mapping on \( \mathbb{C} \)) mapping \( \infty \) to 1, 0 to \(-1\), and \(-1\) to 0.

a) i) \[ \lim_{z \to \infty} \frac{z^5 + 42z}{z^5 + 42z^7} = \infty \leftrightarrow \]

\[ \lim_{z \to 0} \left( \frac{\left(\frac{1}{z}\right)^5}{\left(\frac{1}{z}\right)^5 + 42\left(\frac{1}{z}\right)} \right) = 0. \]

\[ \frac{1/z^5 + 42/z^7}{1/z^7} \]

\[ = \frac{z^2 + 42z^6}{z^5} \to 0 \text{ as } z \to 0 = \text{RHS}. \]

b) Let the transform be \( z \mapsto \frac{az + b}{cz + d} = T(z). \)

\[ T(\infty) = 1 \Rightarrow \frac{a}{c} = 1 \text{ (via a)i)} \]
\[ T(0) = -1 \Rightarrow \frac{b}{d} = -1 \text{ (i)} \]
\[ T(-1) = 0 \Rightarrow a - b = 0 \text{ (i)} \]

So taking \( a = 1, b = 1, c = 1, d = -1 \)

\[ T(z) = \frac{z + 1}{z - 1} \text{ check!} \]

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