6. Linear Transformations

Let $V, W$ be vector spaces over a field $\mathbb{F}$. A function that maps $V$ into $W$, $T : V \to W$, is called a **linear transformation** from $V$ to $W$ if for all vectors $u$ and $v$ in $V$ and all scalars $c \in \mathbb{F}$

(a) $T(u + v) = T(u) + T(v)$

(b) $T(cu) = cT(u)$

**Basic Properties of Linear Transformations**

Let $T : V \to W$ be a function.

(a) If $T$ is linear, then $T(0) = 0$

(b) $T$ is linear if and only if $T(av + w) = aT(v) + T(w)$ for all $v, w$ in $V$ and $a \in \mathbb{F}$. 

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In the special case where $V = W$, the linear transformation $T : V \to V$ is called a linear operator on $V$.

Examples

1. $T : \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $T(a, b) = (2a + b, a)$

2. $T : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ s.t. $T(A) = A^T$

3. $T : P_n(\mathbb{R}) \to P_{n-1}(\mathbb{R})$ s.t.
   $T(f(x)) = f'(x)$

4. $C(\mathbb{R})$ is the space of cts real valued functions on $\mathbb{R}$. Fix $a, b \in \mathbb{R}$ s.t. $a < b$. Then
   
   $T : C(\mathbb{R}) \to \mathbb{R}$ s.t. $T(f) = \int_a^b f(t) \, dt$.

5. Identity operator: For any $V$, $I : V \to V$ s.t. $I(x) = x$

6. Zero transformation: For any $V, W$, $T_0 : V \to W$ s.t. $T_0(x) = 0$
Kernel and Image

Definitions

Let \( T : V \rightarrow W \) be a linear transformation.

The set of vectors in \( V \) that \( T \) maps into \( 0 \) is called the \textbf{kernel} of \( T \). It is denoted by \( \text{ker}(T) \). In mathematical notation:

\[
\text{ker}(T) = \{ v \in V | T(v) = 0 \}
\]

The set of all vectors in \( W \) that are images under \( T \) of at least one vector in \( V \) is called the \textbf{Image} of \( T \); it is denoted by \( \text{Im}(T) \). In mathematical notation:

\[
\text{Im}(T) = \{ w \in W | w = T(v) \text{ for some } v \in V \}
\]

Theorem

Let \( T : V \rightarrow W \) be linear. Then \( \text{ker}(T) \) and \( \text{Im}(T) \) are subspaces of \( V \) and \( W \) respectively.
Example

\[ T : \mathbb{R}^3 \to \mathbb{R}^2 \text{ s.t. } T(a, b, c) = (a - b, 2c) \]
**Theorem**

If $T : V \rightarrow W$ is a linear transformation and \{${v_1, v_2, \ldots, v_n}$\} forms a basis for $V$, then $\text{Im}(T) = \text{span}(T(v_1), T(v_2), \ldots, T(v_n))$
**Rank and Nullity**

**Definitions** If \( T : U \to V \) is a linear transformation,

- the dimension of the image of \( T \) is called the **rank of** \( T \) and is denoted by rank(\( T \)),

- the dimension of the kernel is called the **nullity of** \( T \) and is denoted by nullity(\( T \)).

**Example**

Let \( U \) be a vector space of dimension \( n \), with basis \( \{u_1, u_2, \ldots, u_n\} \), and let \( T : U \to U \) be a linear operator defined by

\[
T(u_i) = u_{i+1}, \quad i = 1, \ldots, n-1, \quad T(u_n) = 0
\]

Find bases for \( \ker(T) \) and \( \operatorname{Im}(T) \) and determine rank(\( T \)) and nullity(\( T \)).
Theorem

If $T : V \rightarrow W$ is a linear transformation from an $n$-dimensional vector space $V$ to a vector space $W$, then

$$\text{rank}(T) + \text{nullity}(T) = \text{dim}(V) = n$$
**Theorem**

Let $T : V \rightarrow W$ be linear. Then $T$ is injective if and only if $\ker(T) = \{0\}$.

**Theorem**

Let $T : V \rightarrow W$ be linear and $\dim(V) = \dim(W)$. Then the following are equivalent:

- $T$ is injective
- $T$ is surjective
- $\text{rank}(T) = \dim(V)$
Theorem

Suppose that \( \{v_1, v_2, \ldots, v_n\} \) is a basis for \( V \). For \( w_1, w_2, \ldots, w_n \) in \( W \) there exists exactly one linear transformation \( T : V \rightarrow W \) such that \( T(v_i) = w_i, i = 1, 2, \ldots, n. \)
Corollary

Let $\{v_1, v_2, \ldots, v_n\}$ be a basis for $V$ and let $T_1, T_2 : V \to W$ be linear s.t. $T_1(v_i) = T_2(v_i)$ for $i = 1, 2, \ldots, n$. Then $T_1 = T_2$. 
Example

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ s.t. $T(a, b, c) = (a - b, 2c)$. Suppose $U : \mathbb{R}^3 \to \mathbb{R}^2$ is linear and

$U(1, 1, 1) = (0, 2), \quad U(1, 0, -1) = (1, -2), \quad U(0, -1, 1) = (1, -2)$. 
Quiz
True or false?

• If $T(x + y) = T(x) + T(y)$ then $T$ is linear.

• If $T : V \to W$ is linear then $T(0_V) = 0_W$.

• $T$ is injective if and only if the only vector $x$ satisfying $T(x) = 0$ is $x = 0$.

• Given $x_1, x_2 \in V$ and $y_1, y_2 \in W$, there exists a linear transformation $T : V \to W$ s.t. $T(x_1) = y_1$ and $T(x_2) = y_2$. 