5. Inner product spaces and orthonormal bases

We only consider $F = \mathbb{R}$ or $\mathbb{C}$.

Definition

Let $V$ be a vector space over $F$. We define an inner product $\langle \cdot, \cdot \rangle$ on $V$ to be a function that assigns a scalar $\langle u, v \rangle \in F$ to every pair of ordered vectors $u, v \in V$ such that the following properties hold for all $u, v, w \in V$ and $\alpha \in F$:

(a) $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$

(b) $\langle \alpha u, v \rangle = \alpha \langle u, v \rangle$

(c) $\langle u, v \rangle = \langle v, u \rangle$

(d) $\langle u, u \rangle > 0$ if $u \neq 0$. 

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Examples

1. \( V = \mathbb{F}^n \), \( \langle \mathbf{u}, \mathbf{v} \rangle \equiv \mathbf{u} \cdot \mathbf{v} \) determined by
   \[
   \mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{n} u_i v_i,
   \]
   \( \mathbf{u} = (u_1, u_2, \ldots, u_n) \) and \( \mathbf{v} = (v_1, v_2, \ldots, v_n) \).

2. Define \( \langle A, B \rangle \) on \( M_n(\mathbb{F}) \) by
   \[
   \langle A, B \rangle = \text{tr}(B^* A).
   \]
Theorem

Let $V$ be an inner product space. For $x, y, z \in V$ and $c \in \mathbb{F}$, we have

(a) $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$

(b) $\langle x, cy \rangle = c \langle x, y \rangle$

(c) $\langle x, 0 \rangle = \langle 0, x \rangle = 0$

(d) $\langle x, x \rangle = 0$ iff $x = 0$

(e) If $\langle x, y \rangle = \langle x, z \rangle \forall x \in V$, then $y = z$. 

Definitions

• A vector space $V$ over $F$ endowed with a specific inner product is called an inner product space. If $F = \mathbb{R}$ then $V$ is said to be a real inner product space, whereas if $F = \mathbb{C}$ we call $V$ a complex inner product space.

• The norm (or length, or magnitude) of a vector $u$ is given by $\|u\| = \sqrt{\langle u, u \rangle}$.
• Two vectors $u, v$ in an inner product space are said to be **orthogonal** if $\langle u, v \rangle = 0$.

• If $u$ and $v$ are orthogonal vectors and both $u$ and $v$ have a magnitude of one (with respect to $\langle , \rangle$), then $u$ and $v$ are said to be **orthonormal**.

• A set of vectors in an inner product space is called an **orthogonal set** if all pairs of distinct vectors in the set are orthogonal. An orthogonal set in which each vector has a magnitude of one is called an **orthonormal set**.
Theorem

Every non-zero finite dimensional inner product space $V$ has an orthonormal basis.
Example: Consider the vector space $\mathbb{R}^3$ with the Euclidean inner product. Apply the Gram-Schmidt process to transform the basis vectors $u_1 = (1, 1, 1), u_2 = (0, 1, 1), u_3 = (0, 0, 1)$ into an orthogonal basis $\{v_1, v_2, v_3\}$; then normalize the orthogonal basis vectors to obtain an orthonormal basis $\{q_1, q_2, q_3\}$.
Quiz
True or false?

• An inner product is a scalar-valued function on the set of ordered pairs of vectors.

• An inner product space must be over the field of real or complex numbers.

• An inner product is linear in both components.

• If $x$, $y$ and $z$ are vectors in an inner product space such that $\langle x, y \rangle = \langle x, z \rangle$, then $y = z$.

• If $\langle x, y \rangle = 0$ for all $x$ in an inner product space, then $y = 0$. 