3. Span and linear dependence

Definitions

• A vector $w$ is called a **linear combination** of the vectors $v_1, v_2, \ldots, v_r$ if it can be expressed in the form

$$w = k_1v_1 + k_2v_2 + \cdots + k_rv_r$$

where $k_1, k_2, \ldots, k_r$ are scalars.

• For a set $S$ of vectors in a vector space $V$, the **span** of $S$ (denoted $span(S)$) is the set consisting of all linear combinations of the vectors in $S$. 
Theorem

If \( S = \{v_1, v_2, \ldots, v_r\} \) is a set of vectors in a vector space \( V \), then:

(a) \( \text{span}(S) \) is a subspace of \( V \).

(b) \( \text{span}(S) \) is the smallest subspace of \( V \) that contains \( v_1, v_2, \ldots, v_r \) and every other subspace of \( V \) that contains \( v_1, v_2, \ldots, v_r \) must contain \( \text{span}(S) \).

We say that a subset \( S \) of a vector space \( V \) spans \( V \) if \( \text{span}(S) = V \).
Example

The polynomials $1, x, x^2, \ldots, x^n$ span the vector space $P_n$ defined previously since each polynomial $p$ in $P_n$ can be written as

$$p = a_0 + a_1x + \cdots + a_nx^n$$

which is a linear combination of $1, x, x^2, \ldots, x^n$. This can be denoted by writing

$$P_n = span\{1, x, x^2, \ldots, x^n\}$$
Definition

If $S = \{v_1, v_2, \ldots, v_r\}$ is a nonempty set of vectors, then the vector equation

$$k_1 v_1 + k_2 v_2 + \cdots + k_r v_r = 0$$

has at least one solution, namely

$$k_1 = 0, k_2 = 0, \ldots, k_r = 0$$

If this is the only solution, then $S$ is called a **linearly independent** set. If there are other solutions, then $S$ is called a **linearly dependent** set.
Examples

1. If \( v_1 = (2, -1, 0, 3), v_2 = (1, 2, 5, -1) \) and \( v_3 = (7, -1, 5, 8) \), then the set of vectors \( S = \{ v_1, v_2, v_3 \} \) is linearly dependent, since \( 3v_1 + v_2 - v_3 = 0 \).

2. The polynomials

\[ p_1 = 1 - x, \quad p_2 = 5 + 3x - 2x^2, \quad p_3 = 1 + 3x - x^2 \]

form a linearly dependent set in \( P_2 \) since

\[ 3p_1 - p_2 + 2p_3 = 0 \]
3. Consider the vectors $i = (1, 0, 0), j = (0, 1, 0), k = (0, 0, 1)$ in $\mathbb{R}^3$. In terms of components the vector equation

$$k_1 i + k_2 j + k_3 k = 0$$

becomes

$$k_1(1, 0, 0) + k_2(0, 1, 0) + k_3(0, 0, 1) = (0, 0, 0)$$

or equivalently,

$$(k_1, k_2, k_3) = (0, 0, 0)$$

Thus the set $S = \{i, j, k\}$ is linearly independent. A similar argument can be used to extend $S$ to a linear independent set in $\mathbb{R}^n$. 
4. In $M_{2 \times 3}(\mathbb{R})$, the set

$$\left\{ \begin{pmatrix} 1 & -3 & 2 \\ -4 & 0 & 5 \end{pmatrix}, \begin{pmatrix} -3 & 7 & 4 \\ 6 & -2 & -7 \end{pmatrix}, \begin{pmatrix} -2 & 3 & 11 \\ -1 & -3 & 2 \end{pmatrix} \right\}$$

is linearly dependent since

$$5 \begin{pmatrix} 1 & -3 & 2 \\ -4 & 0 & 5 \end{pmatrix} + 3 \begin{pmatrix} -3 & 7 & 4 \\ 6 & -2 & -7 \end{pmatrix} - 2 \begin{pmatrix} -2 & 3 & 11 \\ -1 & -3 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$
Theorem

Let $S_1$ and $S_2$ be subsets of a vector space such that $S_1 \subseteq S_2$.

- If $S_1$ is linearly dependent, then so is $S_2$.
- IF $S_2$ is linearly independent, then so is $S_1$. 
Theorem

Let $S$ be a linearly independent subset of a vector space $V$, $v \in V$ and $v \notin S$. Then $S \cup \{v\}$ is linearly dependent iff $v \in \text{span}(S)$. 
Quiz
True or false?

(a) 0 is a linear combination of any non-empty set of vectors.

(b) If $S \subseteq V$ (vector space $V$), then $\text{span}(S)$ equals the intersection of all subspaces of $V$ that contain $S$.

(c) If $S$ is a linearly independent set, then each vector in $S$ is a linear combination of other vectors in $S$.

(d) Any set of vectors containing the zero vector is linearly dependent.