1. Find all the critical points of the following systems. Derive the linearized system about each critical point and use this to classify the critical points as to type (node, saddle, center, spiral) and stability.

a) \[
\begin{pmatrix}
\dot{y}_1 \\
\dot{y}_2
\end{pmatrix}
= \begin{pmatrix}
2y_1 + y_2 \\
-5y_1 + 5
\end{pmatrix}
\]
b) \[
\begin{pmatrix}
\dot{y}_1 \\
\dot{y}_2
\end{pmatrix}
= \begin{pmatrix}
3y_1 + y_1y_2 \\
y_1 + y_2 - y_2^2
\end{pmatrix}
\]
c) \[
\begin{pmatrix}
\dot{y}_1 \\
\dot{y}_2
\end{pmatrix}
= \begin{pmatrix}
2 + 2y_1 - y_1y_2^2 \\
y_2^2y_1 - 2y_2
\end{pmatrix}
\]
(Hint. There are 3 critical points)

d) A version of Duffings equation \( \ddot{x} - x + x^3 = 0 \) let \( y_1 = x \) and \( y_2 = \dot{x} \)
e) A damped version of Duffings equation \( \ddot{x} - 0.1\dot{x} + x^3 = 0 \) let \( y_1 = x \) and \( y_2 = \dot{x} \)

2*. Typically overcrowding of a population inhibits growth which can be modelled by a negative quadratic term in the rate equation. If this occurs for the rabbits in the Lotka-Volterra Population model the result is a more complicated model of the following form.

\[
\dot{r} = r - r^2 + rf \\
\dot{f} = -rf + 3f
\]

a) Derive all the critical points of this system. (There are three.)
b) Find the linearized system about each critical point, classify each as to type (node, saddle, center, spiral) and stability.
c) Sketch the phase portrait of each critical point in the full \((r, f)\) space.
For a node or saddle you need to find and sketch the straight line solutions and then sketch a couple of the other trajectories.
For a center or spiral find the orientation and stability and then sketch a few trajectories.
Don’t forget to include arrows indicating the direction of time.

Supposing there are no other known structures (limit cycles say) could you join up the tractors of your diagram? If the initial rabbit and fox population were nonzero what would their final (scaled so 1 rabbit might mean 100) population be?