MATH2100 Problem Set 10  
( = MATH2011 Problem Set 5)

1: A long cylinder of material with a square cross-section occupies the region $0 < x < a$, $0 < y < a$ in space. The faces at $x = 0$, $x = a$ and $y = 0$ are held at temperature $u = 0$, and the face at $y = a$ is held at temperature $u = u_0$ (const.) until a steady temperature $u(x, y)$ is reached in the cylinder. Show that

$$u(x, y) = \frac{4u_0}{\pi} \sum_{m=1}^{\infty} \frac{\sin[(2m - 1)\pi x/a] \sinh[(2m - 1)\pi x/a]}{(2m - 1) \sinh[(2m - 1)\pi]},$$

for $0 < x < a$, $0 < y < a$.

Hence deduce that on the central axis of the cylinder,

$$u(a/2, a/2) = \frac{2u_0}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^m}{(2m - 1) \cosh[(2m - 1)\pi/2]}.$$

(Hint: Use $\sinh(2z) = 2 \sinh(z) \cosh(z).$)

By considering the steady-state problem when all four faces of the cylinder are held at temperature $u_0$, deduce that it must also be true that $u(a/2, a/2) = u_0/4$, and hence that

$$\sum_{m=1}^{\infty} \frac{(-1)^m}{(2m - 1) \cosh[(2m - 1)\pi/2]} = \frac{\pi}{8}.$$

2: Show that $u(x, y) = 2x + 3xy^2 - 3x^2y - x^3 + y^3$ is a harmonic function, and find a conjugate harmonic function $v(x, y)$.

Check that $\nabla u \cdot \nabla v = 0$ holds for all $x$ and $y$.

3: K Set 11.5 p. 609 No. 17.

4: K Set 11.5 p. 609 No. 19.

Solutions to problems 1 and 2 to be handed in at end of tutorial on Thursday, October 21 or Friday, October 22, or in box on Level 4, Bldg 67, by 10 am on Monday, October 25.