1. a) Critical points are where
\[
\begin{pmatrix}
\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = (f_1(y_1, y_2) , f_2(y_1, y_2)) \\
\end{pmatrix}
\]
\[
f_1(y_1, y_2) = 0 \text{ AND } f_2(y_1, y_2) = 0.
\]
a) Here this gives
\[2y_1 + y_2 = 0 \text{ and } -5y_1 + 5 = 0 \Rightarrow y_1 = 1 \text{ and } y_2 = -2 \]
\[(1, -2) \text{ is a critical pt. (This is a linear system so there is only one)}\]
The linearized System about \((y_1^*, y_2^*)\) is
\[
\begin{pmatrix}
\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \left( \begin{array}{cc}
\frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\
\frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \\
\end{array} \right) \begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \end{bmatrix} \end{pmatrix}
\]
with the matrix \(Df\) evaluated at \((y_1^*, y_2^*)\)
\[
Df = \left( \begin{array}{cc}
\frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\
\frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \\
\end{array} \right) = \left( \begin{array}{cc}
2 & 1 \\
-5 & 0 \\
\end{array} \right)
\]
Now \[\text{det } Df = 5 > 0 \text{ and } \text{trace } Df = 2 > 0\]
\[\text{trace}^2 - 4 \text{det} = 4 - 4 \times 5 < 0 \Rightarrow \text{unstable spiral or focus}\]

b) Critical points
\[3y_1 + y_1y_2 = 0 \Rightarrow y_1(3 + y_2) = 0 \Rightarrow y_1 = 0 \text{ or } y_2 = -3\]
\[y_1 + y_1y_2 - y_2^2 = 0\]
given \( (0, 0) \) and \((12, -3)\).
\[
Df = \left( \begin{array}{cc}
3 + y_2 & y_1 \\
1 & 1 - 2y_2 \\
\end{array} \right)
\]
At \((0, 0)\) \[Df(0, 0) = \left( \begin{array}{cc}
3 & 0 \\
1 & 1 \\
\end{array} \right) \text{ det } Df = -3\]
\[\text{trace}^2 - 4 \text{det} = 16 - 4 \times 3 > 0 \Rightarrow \text{UNSTABLE NODE}\]
At \((12, -3)\) \[Df(12, -3) = \left( \begin{array}{cc}
0 & 12 \\
1 & 7 \\
\end{array} \right) \text{ det } Df = -12\]
\[\Rightarrow \text{SADDLE UNSTABLE}\]
1. a) \[
\begin{pmatrix}
\dot{y}_1 \\
\dot{y}_2
\end{pmatrix} = \begin{pmatrix} 2 & 1 \\
2 & 3
\end{pmatrix} \begin{pmatrix} y_1 \\
y_2
\end{pmatrix} + \begin{pmatrix} \sin t \\
0
\end{pmatrix} = A \dot{y} + \mathbf{g}
\]

To find the matrix \(X\) which diagonalizes \(A\) first find the eigenvalues \(\lambda\) and vectors of \(A\).

\[\det (A - \lambda \mathbf{I}) = \det \begin{bmatrix} 2 - \lambda & 1 \\
2 & 3 - \lambda \end{bmatrix} = (2 - \lambda)(3 - \lambda) - 2 = \lambda^2 - 5\lambda + 4 = (\lambda - 1)(\lambda - 4)\]

\(\Rightarrow \lambda_1 = 1, \quad \lambda_2 = 4\)

Eigen vectors \(\lambda_1 = 1\) \((A - \lambda_1 \mathbf{I})\mathbf{x} = 0 \Rightarrow \begin{pmatrix} 1 & 1 \\
2 & 2
\end{pmatrix} \begin{pmatrix} x \\\nv
\end{pmatrix} = \begin{pmatrix} 0 \\
0
\end{pmatrix} \Rightarrow \mathbf{x}^{(1)} = \begin{pmatrix} 0 \\
1
\end{pmatrix}\)

\(\lambda_2 = 4\) \((A - \lambda_2 \mathbf{I})\mathbf{x} = 0 \Rightarrow \begin{pmatrix} -2 & 1 \\
2 & -1
\end{pmatrix} \begin{pmatrix} x \\
v
\end{pmatrix} = \begin{pmatrix} 0 \\
0
\end{pmatrix} \Rightarrow \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\
2
\end{pmatrix}\)

So \(X = \begin{pmatrix} 0 & 1 \\
-1 & 2
\end{pmatrix} = \begin{pmatrix} \mathbf{x}^{(1)} & \mathbf{x}^{(2)} \end{pmatrix}\)

This matrix diagonalizes \(A\) so that

\[X^{-1}AX = \begin{pmatrix} 1 & 0 \\
0 & 4\end{pmatrix}, \quad (\text{You can check } X^{-1} = \frac{1}{3} \begin{pmatrix} 2 & -1 \\
1 & 1
\end{pmatrix})
\]

\[X^{-1}AX = \frac{1}{3} \begin{pmatrix} 2 & -1 \\
1 & 1
\end{pmatrix} \begin{pmatrix} 1 & 0 \\
0 & 4
\end{pmatrix} \begin{pmatrix} 0 & 1 \\
-1 & 2
\end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 & 1 \\
0 & 2
\end{pmatrix} = \begin{pmatrix} 1 & 0 \\
0 & 4
\end{pmatrix}\]

Now let \(y = Xz\) and change to new variables \(z\)

\[\dot{y} = X\dot{z} = AXz + \mathbf{g} \Rightarrow \dot{z} = X^{-1}AXz + X^{-1}\mathbf{g} = Dz + \mathbf{h}\]

where \(\mathbf{h} = X^{-1}\mathbf{g} = \frac{1}{3} \begin{pmatrix} 2 & -1 \\
1 & 1
\end{pmatrix} \begin{pmatrix} \sin t \\
0
\end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2\sin t \\
\sin t
\end{pmatrix}\)

Now in component form this gives two first order uncoupled linear ODE's

\[\dot{z}_1 = z_1 + \frac{2}{3} \sin t \quad \text{and} \quad \dot{z}_2 = 4z_2 + \frac{1}{3} \sin t\]

\[\Rightarrow \frac{d}{dt}(z, e^t) = \frac{2}{3} e^t \sin t\]

\[\Rightarrow z_1 = e^t \int \frac{2}{3} e^{-t} \sin t \, dt\]
1a) Now we can integrate $\int e^{-t}\sin t\,dt$ using integration by parts.

Let $I = \int e^{-t}\sin t\,dt$

$$u = \sin t \quad dv = e^{-t}dt$$
$$du = \cos t\,dt \quad v = -e^{-t}$$

$$I = -e^{-t}\sin t + \int e^{-t}\cos t\,dt$$

$$= -e^{-t}\sin t - e^{-t}\cos t - \int e^{-t}\sin t\,dt$$

$$\Rightarrow 2I = -e^{-t}(\sin t + \cos t)$$

$$\Rightarrow I = -\frac{1}{2}e^{-t}(\sin t + \cos t) \quad \text{or use the formula}$$

So $z_1 = \frac{2}{3}e^{t}( -\frac{1}{2}e^{-t}(\sin t + \cos t) + c_1)$

$$= \bar{c}_1 e^{t} - \frac{(\sin t + \cos t)}{3} \quad \bar{c}_1 = \frac{2}{3}c_1$$

Similarly $\frac{d}{dt}(z_2 e^{-4t}) = \frac{1}{3}e^{4t}\sin t$

$$J = \int e^{-4t}\sin t = -\frac{1}{4}e^{-4t}\sin t + \int e^{-4t}\cos t\,dt$$

$$= -\frac{1}{4}e^{-4t}\sin t - \frac{1}{16}e^{-4t}\cos t - \int e^{-4t}\sin t\,dt$$

$$J = -(4\bar{c}_1 e^{-4t}\sin t + \bar{c}_2 e^{-4t}\cos t) \quad \text{or use formula}$$

$$z_2 = \frac{1}{3}e^{4t}( -\frac{1}{4}e^{-4t}(\sin t + \cos t) + c_2)$$

$$= \bar{c}_2 e^{4t} - \frac{1}{4}e^{4t}(4\sin t + \cos t) \quad \bar{c}_2 = \frac{2}{3}c_2$$

Now transform back $y_1 = z_1 + z_2, y_2 = -z_1 + 2z_2$

$$y_1 = \bar{c}_1 e^{t} - \frac{(\sin t + \cos t)}{3} + \bar{c}_2 e^{4t} - \frac{1}{17}e^{4t}(4\sin t + \cos t)$$

$$y_2 = -\bar{c}_1 e^{t} + \frac{(\sin t + \cos t)}{3} + 2\bar{c}_2 e^{4t} - \frac{2}{17}e^{4t}(4\sin t + \cos t)$$
\[ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} t^2 \\ -2 \end{pmatrix} = Ay + g(t) \]

First find eigenvalues and vectors of \( A \)

\[ \lambda^2 - 1 = 0, \quad \lambda = \pm 1 \]

\[ \lambda_1 = 1, \quad (A - \lambda I) \mathbf{x}' = (1, -1) \mathbf{x}' = \mathbf{0} \Rightarrow \mathbf{x}' = (1) \]

\[ \lambda_2 = -1, \quad (1, -1) \mathbf{x}' = \mathbf{0} \Rightarrow \mathbf{x}' = (1) \]

So, \( X = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) implies \( X^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \)

Let \( \mathbf{y} = X \mathbf{z} \Rightarrow \mathbf{z} = D \mathbf{z} + X^{-1} g(t) \)

where \( D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \)

and \( X^{-1} g(t) = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} t^2 \\ -2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} t^2 - 2 \\ t^2 + 2 \end{pmatrix} \)

Now \( * \) gives two decoupled first order linear eqns:

\[ \dot{z}_1 = z_1 + \frac{1}{2} (t^2 - 2) \quad \text{and} \quad \dot{z}_2 = -z_2 + \frac{1}{2} (t^2 + 2) \]

\[ \Rightarrow \frac{d}{dt} (e^t z_1) = \frac{e^t}{2} (t^2 - 2) \quad \text{and} \quad \frac{d}{dt} (e^t z_2) = \frac{1}{2} e^t (t^2 + 2) \]

Now \( \int e^{-t} (t^2 - 2) \, dt = \frac{e^{-t}}{2} (t^2 - 2) - \int e^{-t} 2t \, dt \quad \text{and} \quad \int e^t (t^2 + 2) \, dt = e^t (t^2 + 2) - \int e^t 2t \, dt \)

\[ = \frac{e^{-t}}{2} (t^2 - 2) + e^t (t^2 + 2) + 2 \int e^{-t} \, dt = \frac{e^{-t}}{2} (t^2 + 2) + e^t (t^2 + 2) + C_1 \]

And \( \int e^t (t^2 + 2) \, dt = e^t (t^2 + 2) - \int e^t 2t \, dt \quad \text{and} \quad \int e^{-t} 2t \, dt = \frac{e^{-t}}{2} (t^2 + 2) + C_2 \)

\[ = e^t (t^2 + 2) - e^t 2t + 2e^t + C_2 = e^t (t^2 + 2t + 4) + C_2 \]

So \( z_1 = C_1 e^t - \left( \frac{t^2 + 2t}{2} \right) \quad \text{and} \quad z_2 = C_2 e^t + \left( \frac{t^2 - 2t + 4}{2} \right) \)

Finally \( \mathbf{y} = X \mathbf{z} \Rightarrow y_1 = z_1 + z_2, \quad y_2 = z_1 - z_2 \)

\[ y_1 = C_1 e^t + C_2 e^{-t} - 2t + 2, \quad y_2 = C_1 e^t - C_2 e^{-t} - t^2 - 2 \]
Kneszsig 43.6 p 189

\[ Q(\mathbf{y}) = \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 - 16t^2 \end{pmatrix} = \mathbf{Ay} + g(t) \]

Eigenvalues: \( \lambda^2 - 16 = 0 \) \( \lambda = \pm 4 \).

Vectors:
- \( \mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) and \( \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \)

So \( \mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \) and \( \mathbf{x}^{-1} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \)

Let \( \mathbf{y} = \mathbf{x} \mathbf{z} \Rightarrow \dot{\mathbf{z}} = \mathbf{D} \mathbf{z} + \mathbf{x}^{-1} g(t) \)

where \( \mathbf{D} = \begin{pmatrix} 4 & 0 \\ 0 & -4 \end{pmatrix} \) and \( \mathbf{x}^{-1} g(t) = \begin{pmatrix} 1 \-8t^2 \\ -1 + 8t^2 \end{pmatrix} \)

In component form

\[ \dot{z}_1 = 4z_1 + (1 - 8t^2) \]
\[ \dot{z}_2 = -4z_2 + (-1 + 8t^2) \]

\[ \Rightarrow \frac{d}{dt} (e^{-4t} z_1) = e^{4t} (1 - 8t^2) \]

\[ \Rightarrow z_1 = c_1 e^{4t} + (2t^2 + t) \]

and \( \frac{d}{dt} (e^{4t} z_2) = e^{4t} (-1 + 8t^2) \)

\[ \Rightarrow z_2 = c_2 e^{4t} + 2t^2 + t \]

No transform back \( \mathbf{y} = \mathbf{x} \mathbf{z} \)

\[ \begin{align*}
  y_1 &= z_1 + z_2 \\
  y_2 &= z_1 - z_2 \\
\end{align*} \]

\[ \begin{align*}
  y_1 &= c_1 e^{4t} + c_2 e^{-4t} + 4t^2 \\
  y_2 &= c_1 e^{4t} - c_2 e^{-4t} + 2t \\
\end{align*} \]

Now \( y_1(0) = 3 \Rightarrow c_1 + c_2 = 3 \)
\( y_2(0) = 1 \Rightarrow c_1 - c_2 = 1 \)

\[ \begin{align*}
  y_1 &= 2e^{4t} + e^{-4t} + 4t^2 \\
  y_2 &= 2e^{4t} - e^{-4t} + 2t \\
\end{align*} \]
Q 2 a) Recall the general result

\[ L(t^n) = \frac{n!}{s^{n+1}} \] and use linearity

i) \[ L(t^3 - 2t) = L(t^3) - 2L(t) \]
\[ = \frac{3!}{s^4} - 2 \frac{2}{s^2} = \frac{6}{s^4} - \frac{2}{s^2} \]
Recall the general result
\[ L(\cos(\alpha t)) = \frac{s}{s^2 + \alpha^2} \] and \[ L(\sin(\alpha t)) = \frac{\alpha}{s^2 + \alpha^2} \]

ii) \[ L(\cos 2t) = \frac{s}{s^2 + 9} \]
Recall the shifting theorem
\[ L(e^{-\alpha t} f(t)) = F(s + \alpha) \]
iii) If \[ f(t) = 5 \sin 5t \]
\[ L(f(t)) = \frac{5}{s^2 + 25} \]
\[ L(e^{2t} f(t)) = \frac{5}{(s-2)^2 + 25} \]

iv) First use double angle formulae \[ \cos 2t = (\cos 6t + 1) \]
\[ L(e^{2t} (\cos 6t + 1)) = \frac{2}{2} L(e^{2t} \cos 6t) + \frac{1}{2} L(e^{2t}) \]
\[ = \frac{1}{2} \left( \frac{s-2}{(s-2)^2 + 36} + \frac{1}{s-2} \right) \]
\[ = \frac{(s-2)^2 + 18}{(s-2)(s-2)^2 + 36} \]

v) \[ f(t) = (t^2 + 2t + 4) e^{3t} \]
\[ L(f(t)) = \frac{2!}{(s-3)^3} + \frac{2}{(s-3)^2} + \frac{4}{(s-3)} \]
vi) \[ f(t) = \begin{cases} 2 - t & 0 \leq t \leq 1 \\ 1 & 1 < t \end{cases} \]

\[
L(f(t)) = \int_0^\infty e^{-st} f(t) \, dt = \int_0^1 e^{-st} (2-t) \, dt + \int_1^\infty e^{-st} \, dt
\]

Now,

\[
\int_0^1 e^{-st} (2-t) \, dt = \frac{e^{-st}}{-s} (2-t) \bigg|_0^1 - \int_0^1 e^{-st} \, dt = \frac{e^{-s}}{-s} (2-t) - \frac{1}{s^2} + C
\]

\[
\int_0^1 e^{-st} \, dt = \left( \frac{e^{-st}}{-s} \right) \bigg|_0^1 = \frac{-s}{s^2} (1-s) - \frac{(1-2s)}{s^2}
\]

\[
\int_1^\infty e^{-st} \, dt = \left( \frac{e^{-st}}{-s} \right) \bigg|_1^\infty = \frac{1}{s^2} e^{-s} - \frac{(1-2s)}{s^2}
\]

\[
L(f(t)) = \frac{e^{-s} - (1-2s)}{s^2} = \frac{e^{-s} + 2s - 1}{s^2}
\]

vii) \[ f(t) = \begin{cases} t/2 & 0 \leq t \leq 2 \\ 0 & 2 < t \end{cases} \]

\[
L(f(t)) = \int_0^2 e^{-st} \frac{t}{2} \, dt = t e^{-st} \bigg|_0^2 - \int_0^2 e^{-st} \frac{dt}{2} = \frac{e^{-st}}{-s} \left( \frac{-t}{2s} - \frac{1}{2s^2} \right) \bigg|_0^2
\]

\[
= e^{-s} \left( \frac{-2}{2s} - \frac{1}{2s^2} \right) - \left( 0 - \frac{1}{2s^2} \right)
\]

\[
= e^{-2s} \left( \frac{-s}{s^2} - \frac{1}{2s^2} \right) - \left( 0 - \frac{1}{2s^2} \right)
\]

\[
= -2s \left( \frac{1+4s}{s^2} + \frac{1}{2s^2} \right)
\]
2b) i) \( F(s) = \frac{s+2}{(s^2-4)} = \frac{1}{s+2} \)

So \( f(t) = e^{-2t} \)

ii) \( F(s) = \frac{s+1}{s^2+4s+5} = \frac{5-2+3}{(s-2)^2+1} \)

Now \( L(e^{-at} \cos \omega t) = \frac{s+a}{(s+a)^2 + \omega^2} \)

\( L(e^{-at} \sin \omega t) = \frac{\omega}{(s+a)^2 + \omega^2} \)

So here \( f(t) = e^{-2t} \cos t + 3e^{-2t} \sin t \)

iii) \( F(s) = \frac{1}{s^4} - \frac{1}{(s+3)^2} \)

Now \( L(t^n) = \left( \frac{n!}{s^{n+1}} \right) \) and from the shifting thm

\( L(e^{-at} t^n) = \frac{n!}{(s+a)^{n+1}} \)

So here \( f(t) = \frac{t^3}{3!} - e^{-3t} \)

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Q6. \( L(e^{t \cosh 3t}) = e^t(e^{3t} + e^{-3t}) = \frac{e^{4t} + e^{-2t}}{2} \)

\( L(e^{t \cosh 3t}) = \frac{1}{2} \left( \frac{1}{s-4} + \frac{1}{s+2} \right) \)

\( = \frac{(s-1)}{(s-4)(s+2)} \)
\( \mathbf{K5.1 \ p \ 257} \)

Q12 \( \mathcal{L}(f(t)) = \begin{cases} t & 0 \leq t \leq 1 \\ 2-t & 1 \leq t \leq 2 \\ 0 & 2 \leq t \end{cases} \)

\( \mathcal{L}(f(t)) = \int_{0}^{1} e^{-st} t \, dt + \int_{1}^{2} (2-t) \cdot e^{-st} \, dt \)

Now \( \int e^{-st} \, dt = \frac{e^{-st}}{-s} + C \)

\[ = \frac{e^{-st}}{s^2} (1-s) \left[ \frac{1}{0} + \left( \frac{2e^{-s}}{-s} - \frac{e^{-s}}{s^2} (1-s) \right) \right]^2 \]

\[ = \frac{-e^{-s}}{s^2} (1-s) - \frac{1}{s^2} + \left( \frac{2e^{-2s}}{-s} - \frac{e^{-2s}}{s^2} (1-2s) \right) \]

\[ = \frac{-e^{-s}}{s^2} + \frac{e^{-s}}{s^2} (1-s + 2s + (1-s)) - \frac{1}{s^2} \]

\[ = -
\]

Q22. \( F(s) = \frac{60}{s^7} + \frac{6}{s^5} + \frac{1}{s^3} \)

\( f(t) = \frac{60t^6}{6!} + \frac{6t^4}{4!} + \frac{1t^2}{2!} = \frac{t^6}{12} + \frac{t^4}{4} + \frac{t^2}{2} \)

Q36. \( F(s) = \frac{12}{(s-3)^4} \)

\( f(t) = \frac{12e^{3t}}{3!} t^3 = 2e^{3t} t^2 \)