SUMMARY Of 6 Types of LINEAR PHASE PORTRAITS in 2D

It all depends what the type of eigenvalues $A$ has.

1 **SADDLE** One positive eigenvalue and one negative eigenvalue.

2 **IMPROPER NODE** Two different positive eigenvalues **UNSTABLE** or two different negative eigenvalues **STABLE**.

3 **PROPER NODE** Equal eigenvalues and two linearly independent eigenvectors.

4 **DEGENERATE or INFLECTED NODE** Equal eigenvalues, but one corresponding eigenvector.

5 **SPIRAL or FOCUS** Two complex eigenvalues with either negative real part **STABLE** or positive real part **UNSTABLE**.

6 **CENTER** Two pure imaginary eigenvalues.
4 Critical Points and Stability

In all the systems we have looked at so far (linear, constant coefficient, homogeneous 2-d systems) the origin has been one trajectory all on its own, because if you start at the origin $y = 0$ you stay there.

If $\dot{y} = Ay$ then $y = 0 \implies \dot{y} = 0$.

It is the so called trivial solution we mentioned before.

Infact any point in Phase Space where $\dot{y} = 0$ must be stationary and it is often called a stationary, equilibrium or critical point.

**DEFN**  A 2-dimensional system

$$
\begin{pmatrix}
\dot{y}_1 \\
\dot{y}_2
\end{pmatrix} = \begin{pmatrix}
f_1(y_1, y_2) \\
f_2(y_1, y_2)
\end{pmatrix}
$$

has a critical (or stationary) point at $(y_1^*, y_2^*)$ if $f_1(y_1^*, y_2^*) = 0$ AND $f_2(y_1^*, y_2^*) = 0$.

Nonlinear systems may have more than one critical point. But Linear systems of the form $\dot{y} = Ay$ always have one critical point at the origin and unless $\det A = 0$ this is the only one.

**STABILITY OF CRITICAL POINTS**

Trajectories starting close to the origin, may move away, hang around or tend towards the critical point itself. Think of a nonlinear pendulum. There is a critical point where the pendulum hangs vertically down which we might call stable because if we disturb it a little the pendulum doesn’t move very far. But there is another critical point, where the pendulum is vertically above which is unstable because any slight displacement results in the pendulum falling away.

Roughly speaking if every trajectory that starts close to the critical point at 0 stays close to 0 then we say the critical point at 0 is **STABLE**

A critical point that is **not stable** is called **UNSTABLE**.

For instance a stable focus or node is **stable**, in fact it is asymptotically stable because nearby trajectories actually tend to the critical point at the origin.

A center is also **stable**, but not asymptotically.

But a saddle or an unstable focus or node is **unstable**.

One way to classify the types of critical points is via their stability properties.
Stability Criteria for Critical points

It is the eigenvalues of $A$ that determine the stability properties of the critical point at $0$.

$$
det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & \ a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = \lambda^2 - (a_{11} + a_{22})\lambda + \text{det}A.
$$

But the eigenvalues really only depend on the $\text{trace}A = (a_{11} + a_{22})$ and the $\text{det}A$.

Infact

$$
\lambda_{\pm} = \frac{(\text{trace}A) \pm \sqrt{(\text{trace}A)^2 - 4(\text{det}A)}}{2}.
$$

So if $\text{det}A < 0$ the eigenvalues are real and

$$
\lambda_+ > 0 \text{ and } \lambda_- < 0 \implies \text{SADDLE}.
$$

If $\text{det}A > 0$ and $(\text{trace}A)^2 - 4(\text{det}A) > 0$ the eigenvalues are real and have the same sign

if $(\text{trace}A) < 0 \implies \lambda_+ < 0 \implies \text{STABLE improper NODE} $

if $(\text{trace}A) > 0 \implies \lambda_+ > 0 \implies \text{UNSTABLE improper NODE}$

If $\text{det}A > 0$ and $(\text{trace}A)^2 - 4(\text{det}A) < 0$ the eigenvalues are complex,

if $(\text{trace}A) < 0 \implies Re(\lambda_+) < 0 \implies \text{STABLE FOCUS or SPIRAL}$

if $(\text{trace}A) > 0 \implies Re(\lambda_+) > 0 \implies \text{UNSTABLE FOCUS or SPIRAL}$

If $\text{det}A > 0$ and $(\text{trace}A)^2 - 4(\text{det}A) = 0$ the eigenvalues are equal,

if $(\text{trace}A) < 0 \implies \lambda < 0 \implies \text{STABLE proper or degenerate NODE}$

if $(\text{trace}A) > 0 \implies \lambda > 0 \implies \text{UNSTABLE proper or degenerate NODE}$

If $\text{det}A > 0$ and $(\text{trace}A) = 0$ the eigenvalues are pure imaginary, and the critical point at $0$ is a CENTER, which is STABLE.

Finally if $\text{det}A = 0$ one of the eigenvalues is zero and there is a whole line of critical points.
From the stability Chart you can see that

if \( \det A \geq 0 \) and \( (\text{trace}A) \leq 0 \) then the critical point is STABLE.

But if \( \det A < 0 \) or if \( (\det A \geq 0 \) and \( (\text{trace}A) > 0 ) \) then the critical point is UNSTABLE.

Take some examples. If

\[
\begin{pmatrix}
\dot{y}_1 \\
\dot{y}_2
\end{pmatrix} =
\begin{pmatrix}
2 & -1 \\
3 & -2
\end{pmatrix}
\begin{pmatrix}
y_1 \\
y_2
\end{pmatrix}
\]

First check the \( \det A \), \( \det A = 2(-2) - (-1)(3) = 1 \). We need go no further since \( \det A < 0 \implies \) a SADDLE, which is UNSTABLE.

If

\[
\begin{pmatrix}
\dot{y}_1 \\
\dot{y}_2
\end{pmatrix} =
\begin{pmatrix}
2 & 3 \\
2 & 5
\end{pmatrix}
\begin{pmatrix}
y_1 \\
y_2
\end{pmatrix}
\]

Here \( \det A = 10 - 6 > 0 \), so we need to consider the \( \text{trace}A = 7 > 0 \). This means the critical point is either an unstable node or an unstable focus. So we need to consider \( (\text{trace}A)^2 - 4(\det A) = 49 - 16 > 0 \implies \) it is an UNSTABLE NODE.

\[
\begin{pmatrix}
\dot{y}_1 \\
\dot{y}_2
\end{pmatrix} =
\begin{pmatrix}
-2 & 1 \\
-6 & 2
\end{pmatrix}
\begin{pmatrix}
y_1 \\
y_2
\end{pmatrix}
\]

Once again \( \det A = -4 + 6 > 0 \), But \( \text{trace}A = 0 \), giving a CENTER, which is STABLE.