1. If the **eigenvalues of A are real and distinct** then the solution to $\dot{y} = Ay$ is in the form

$$y = c_1 x^{(1)} e^{\lambda_1 t} + c_2 x^{(2)} e^{\lambda_2 t}$$

AND there are two straight lines in the phase portrait associated with the solutions for $c_1 = 0$ and $c_2 = 0$.

2. If the **eigenvalues of A are both negative**

the origin is said to be a stable (improper) **NODE**.

If the **eigenvalues of A are both positive**

the origin is said to be an unstable (improper) **NODE**.

If the **eigenvalues of A are of opposite sign**

the origin is said to be a **SADDLE**

3. The **slope of the curves in phase space** is given by

$$\frac{dy_2}{dy_1} = \frac{\dot{y}_2}{y_1} \quad \text{(from the chain rule.)} = \frac{a_{21} y_1 + a_{22} y_2}{a_{11} y_1 + a_{12} y_2}$$

From this you can tell where the phase curves are horizontal and where they are vertical.

4. The solutions to the equation

$$\frac{dy_2}{dy_1} = \frac{a_{21} y_1 + a_{22} y_2}{a_{11} y_1 + a_{12} y_2}$$

are the phase space curves.
If the eigenvalues are equal there may still be two linearly independent eigenvectors. For instance if
\[
A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]
then \( \lambda = 1 \) and both \( x^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( x^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \)
are eigenvectors.
So the General Solution is
\[
y = \left( c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) e^{t}. \quad \Rightarrow \quad \frac{y_2}{y_1} = \frac{c_2}{c_1}.
\]
So now the trajectories are all straight lines.

This is called a **PROPER NODE**

and it is unstable if \( \lambda > 0 \) and stable if \( \lambda < 0 \).

**Alternatively there may be only one eigenvector.** For instance for
\[
A = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}
\]
then also \( \lambda = 1 \) but there is only one eigenvector \( x^1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \)
Then there is only one straight line \( \frac{y_2}{y_1} = 1 \) in the phase plane.

This is called an **DEGENERATE or INFLECTED NODE** and it is unstable if \( \lambda > 0 \) and stable if \( \lambda < 0 \).

To get some idea as to how the other trajectories come into the origin consider.
\[
\frac{dy_2}{dy_1} = \frac{-y_1 + 2y_2}{y_2}
\]
which is the slope of the trajectory at the point \((y_1, y_2)\)
Along \( y_2 = \frac{1}{2}y_1 \) the trajectories are horizontal, but along \( y_2 = 0 \) they are vertical! The trajectories bend around, almost spiraling, but not quite.

This is called an **DEGENERATE or INFLECTED NODE** and it is unstable if \( \lambda > 0 \) and stable if \( \lambda < 0 \).
Phase Portraits for Complex Eigenvalues

If the eigenvalues are pure imaginary you can prove that the trajectories are ellipses. (See diagonalization later.) Take the following example

\[
\begin{pmatrix}
\dot{y}_1 \\
\dot{y}_2
\end{pmatrix} = \begin{pmatrix}
0 & 1 \\
-2 & 0
\end{pmatrix} \begin{pmatrix}
y_1 \\
y_2
\end{pmatrix}
\]

Then

\[
\frac{dy_2}{dy_1} = \frac{\dot{y}_2}{\dot{y}_1} \implies \frac{dy_2}{dy_1} = -\frac{2y_1}{y_2}
\]

This is separable giving

\[
\int y_2 dy_2 = -2 \int y_1 dy_1 \implies y_2^2 + 2y_1^2 = C \text{ for some constant } C
\]

Ellipses with the $y_2$ axis as the major axis and the $y_1$ axis as the minor.

If say $C = 1$ the ellipse crosses the $y_2$ axis at $\pm 1$ and the $y_1$ axis at $\pm \frac{1}{\sqrt{2}}$. In general the ellipse crosses the $y_2$ axis at $\pm \sqrt{C}$ and the $y_1$ axis at $\pm \frac{1}{\sqrt{2}}$.

Determining the direction of flow.

Go back to the equation of motion for $y_1$; $\dot{y}_1 = y_2$, and consider the sign of $\dot{y}_1$ on the upper part of the $y_2$ axis. On the upper part of the $y_2$ axis $y_1 = 0$ and $y_2 > 0$. So $\dot{y}_1 = y_2 > 0$ and $y_1$ must be increasing. So the motion is clockwise.

This method can also be used to find the direction of flow for spirals.

Any system with pure imaginary eigenvalues is called a CENTER and has elliptical trajectories, but finding the actual elliptic orbit may be tricky (until you know about diagonalization).
If the eigenvalues are complex with nonzero real part then the trajectories spiral in (if the real part is negative) or out (if the real part is positive).

To prove this in general requires diagonalization, but it is easily seen in the following case.

\[
\begin{pmatrix}
\dot{y}_1 \\
\dot{y}_2
\end{pmatrix} = \begin{pmatrix}
2 & -1 \\
1 & 2
\end{pmatrix}
\begin{pmatrix}
y_1 \\
y_2
\end{pmatrix}
\]

which has complex solutions \( \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{(2+i)t} \) and its complex conjugate.

So the real solutions are

\[
\begin{pmatrix}
\cos(t) \\
\sin(t)
\end{pmatrix} e^{2t} \text{ and } \begin{pmatrix}
\sin(t) \\
-\cos(t)
\end{pmatrix} e^{2t}.
\]

For each solution

\[y_1^2 + y_2^2 = e^{4t}\]

which in polar coordinates, \((y_1 = r \cos \theta, y_2 = r \sin \theta)\) is \(r = e^{2t}\).

So solutions spiral out from the origin.

To get some idea of how they do this consider the slope of the trajectories

\[
\frac{dy_2}{dy_1} = \frac{y_1 + 2y_2}{2y_1 - y_2} \implies \text{zero slope on } y_2 = -\frac{y_1}{2} \text{ and infinite slope on } y_2 = 2y_1.
\]

Also the slope is 1 on \(y_2 = \frac{y_1}{3}\) and -1 on \(y_2 = -3y_1\).

Any system with complex eigenvalues is called a SPIRAL.
If the real part of the eigenvalues is positive solutions spiral out from the origin and it is UNSTABLE.
If the real part of the eigenvalues is negative solutions spiral in to the origin and it is STABLE.